University Studies—No. 6.

A DISCUSSION
OF
The Nature of Solutions of Differential Equations of the First Order

LEADING TO A THEORY OF SINGULAR SOLUTIONS
WHICH PRESENTS THE
SINGULAR SOLUTION AS DERIVED IN THE COURSE OF
NATURAL AND LOGICAL STUDY OF
THE COMPLETE PRIMITIVE
WITH
ANALOGIES AND CONTRASTS BETWEEN
DIFFERENTIAL AND DIFFERENCE EQUATIONS

BY
JNANSARAN CHAKRABARTI, M.A.

Calcutta:
PUBLISHED BY THE CALCUTTA UNIVERSITY AND PRINTED
AT THE BAPTIST MISSION PRESS.

1910.

BCV 1717

Gs 3668

CONTENTS.

Page

INTRODUCTION v

ART. 1.—Differential equations and their geometrical interpretation. "Analytical and geometrical solutions" of differential equations. "Perfect and imperfect solutions." Illustrations 1

ART. 2.—Linear differential equations of the first order and degree. Nature of solutions—both analytical and geometrical. Complete primitives and particular solutions. No particular solution excepting those included in the complete primitive 3

ART. 3.—Equations of the first order but not of the first degree, which are resolvable into two or more rational linear equations. Nature of solutions of such equations—geometrical and analytical. Number of "geometrical complete primitives" infinite, but number of analytical complete primitive only one. "Fundamental and derived" geometrical complete primitives and particular solutions 7

ART. 4.—General remarks on the analytical representation of curves. General theorems connected with the properties of curves representable by equations .. 13

ART. 5.—Reason why a resolvable differential equation of the first order, but not of the first degree, has an infinite number of derived "geometrical complete primitives" but not more than one "perfect complete primitive." 16

ART. 6.—Differential equations irresolvable into linear and rational equations. Essential analogy and contrast between the resolvable and irresolvable equations :—(1) As regards mode of solution. Illustrations. General theorem, *viz.*, If $F(x, y, c)=0$ is the complete primitive of $\Phi(x, y, p)=0$ then the component equations of F containing irrational and imaginary expressions are solutions of the corresponding component equations of Φ, F and ϕ being solved with regard to c and p respectively 18

ART. 7.—(2) As regards geometrical interpretation. Though an irresolvable differential equation of the n^{th} degree represents an intersecting system of curves such that n of them pass through a point, the system *may* be looked upon as n sub-systems of non-intersecting curves Illustrations 29

Page.

Art 8.—(3) As regards analytically expressible "derived geometrical particular solutions." Of all the derived geometrical particular solutions one may be "perfect" in the case of irresolvable equations. This is called the singular solution and is generally the envelope of the system of curves making up the geometrical complete primitive. In some cases a singular solution exists even when the geometrical complete primitive has no envelope properly speaking. Hence it is better to use the term "contact curve" than to use the term envelope in connection with singular solutions .. 37

Art. 9.—Analogy and contrast with the calculus of finite differences. "Geometrical and analytical particular solutions" and "complete primitives" of difference equations. Nature of solutions of rational linear difference equations of the first order the same as in differential equations. In the case of equations of higher degrees also, "derived geometrical particular solutions" and complete primitives formed exactly as in differential equations. But whereas in differential equations no derived "geometrical complete primitive" and none or only one of the derived "geometrical particular solutions" is "perfect," in the case of difference equations an infinite number of "geometrical complete primitives" and consequently a doubly infinite number of "derived geometrical particular solutions" are so. Illustrations 42

(*N.B.*—Words coined for this paper or used in a special sense are within inverted commas in the above.)

INTRODUCTION.

The subject of the present paper is an enquiry into the nature of solutions of differential equations, chiefly with reference to their geometrical interpretation, and the investigation of the connection that exists between the complete primitive and the singular solution. As the history of the subject shows, the singular solution was discovered by accident, and its relation with the complete primitive subsequently made out with the greatest difficulty and divergence of opinion by the mathematicians of the last and present centuries. In the following pages an attempt has been made to show how the singular solution might present itself in the natural and logical course of studying the complete primitive, and an intermediate step has been supplied in the shape of derived geometrical particular solution, through which to pass from the complete primitive to the envelope singular solution.

The examination of the nature of solutions of differential equations has engaged the attention of the most eminent analysts from the end of the seventeenth century. *Leibnitz*, *Taylor*, and *Clairut*, by their scattered discussion of the solutions of various differential equations paved the way for *Euler*, whose famous "Paradoxes on Calcul Integral," published in 1756, is the earliest distinct memoir on the subject of singular solutions of differential equations. *Laplace*,[1] *Lagrange*,[2] and *Cauchy*[3] each formulated his own ideas relating to the nature of singular solutions, and laid down his formulæ (in no case faultless and free from error), for finding, and criteria for distinguishing them. Professor *De Morgan*,[4] after careful examination, adopted Lagrange's results in a slightly modified form, "expressing, however, only a qualified confidence in his method." *Boole*,[5] whose extraordinary logical talent adds special value to all his mathematical discussions, built an elaborate theory of singular solutions on the work of all his predecessors on the subject. The theory of singular solutions as at present accepted was laid down in a brief memoir by *Cayley*[1] and illustrated by *Glaisher*,[2] and developed and elaborated in two remarkable papers by *Workman*[3] and Miss *Maddison*.[4] The writings of *Mukhopadhyay*[5] on the subject of differential equation throw much important light on the nature of solutions of such equations.

---

[1] Memoires de L'Academie des Sciences, 1772.

[2] Calcul des Fonctions Leçons, xiv—xvii.

[3] Moigno, Calcul, vol. ii, p. 435.

[4] Cambridge Phil. Transactions, vol. ix, p. ii.

[5] Differential Equations, chapter viii. Supplementary volume, chapter xxi. Calculus of Finite Differences.

It may be here mentioned that Miss Maddison has prefixed to her paper above referred to a complete bibliography of the subject, which is invaluable to those who desire to enter into the study of this interesting branch of mathematics, historically or otherwise.

---

[1] Messenger of Mathematics, vol. ii (872), pp. 6-12.

[2] Messenger of Mathematics, vol. xii (1882), pp. 1-14.

[3] Quarterly Journal of Pure and Mixed Mathematics, vol. xxii (1887).

[4] Quarterly Journal of Pure and Mixed Mathematics, vol. xxvii (1896).

[5] Journal of the Asiatic Society of Bengal, vol. lvi, part ii, 1887.

A Discussion of the Nature of Solutions of Differential Equations of the First Order.

Article 1.

A differential equation of the first order is a relation between x, y and $\frac{dy}{dx}$. Any relation between x and y is said to be a solution of the equation when, on differentiation and algebraical transformations with the help of the original relation it leads to the same value of $\frac{dy}{dx}$ as the given differential equation.

A differential equation of the first order is also capable of a geometrical interpretation. If x and y are the abcissa and ordinate of any point on a curve, $\frac{dy}{dx}$ gives the direction of the curve at that point. A differential equation of the first order $\frac{dy}{dx}=f(x, y)$ accordingly gives a direction for every point in the co-ordinate plane. And a point would be describing a curve according to the differential equation if at any point on the plane the direction of the curve is that which is given by the differential equation.

Such a curve described according to the differential equation may be called a *geometrical solution* of the differential equation. In contradistinction, any functional relation between x and y leading to the same value of $\frac{dy}{dx}$ as the differential equation may be called an *analytical solution*. When a curve is a geometrical solution of a differential equation, and the equation of the curve in x and y is found to be an analytical solution of it, then the curve and its equation may be said to constitute a *perfect solution* of the differential equation. On the other hand when there is a geometrical solution without a corresponding relation between x and y satisfying the differential equation, or an analytical solution without a corresponding locus which is a geometrical solution of the differential equation, then these may be said to be *imperfect* solutions.

The fact that every curve which may be drawn is not necessarily one which is capable of being represented by an equation, as also the fact that every equation in x and y (*e.g.*, one containing imaginary quantities) has not a corresponding graph, make the existence of imperfect solutions possible.

Thus we know that the differential equation of the first order $\frac{dy}{dx}=f(x, y)$ is not always integrable. Hence the analytical solution cannot be always arrived at. But as the differential equation gives a direction for every point in the co-ordinate plane, any number of geometrical solutions can be very easily constructed. These, however, will be imperfect solutions, since there are no corresponding analytical solution of the equation. For instance $\frac{dy}{dx}= +\sqrt{x^{2n}+y^{2n}}$ is an equation, the analytical solution of which cannot be found. But a geometrical solution is easily constructed and found to consist of two infinite semi-parabolic branches in opposite quadrants (Fig. 1), touching the axis of x at the origin which is a point of inflection (not 3-pointic necessarily) on the curve.

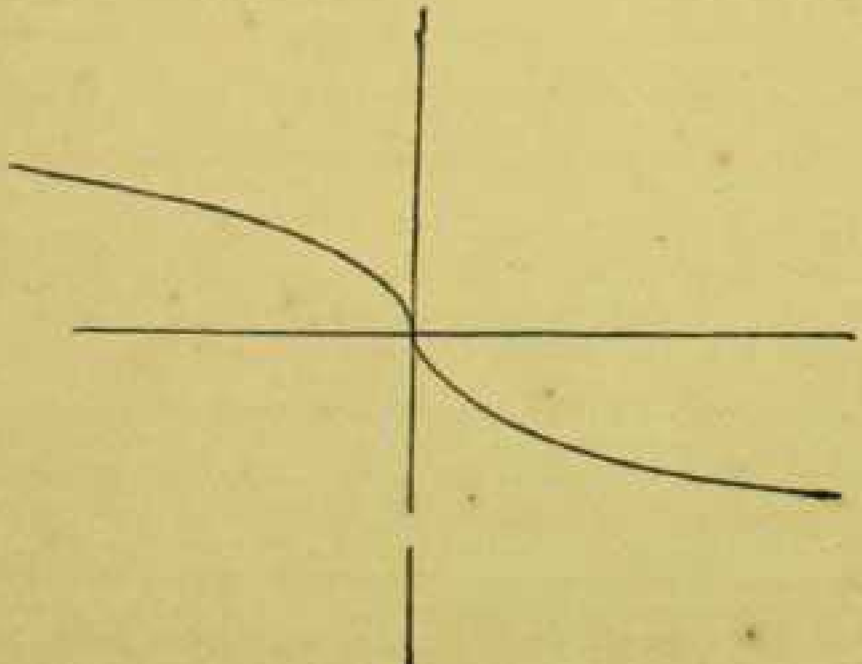

FIG. 1.

Imperfect geometrical solutions of a totally different kind will be discussed more fully later on in this paper. Again, take the differential equation

$$\left(\frac{dy}{dx}\right)^2+1=0.$$

An integral of this is clearly $y=ix$ or $y=-ix$. But we shall call these imperfect analytical solutions inasmuch as there are no corresponding geometrical solutions to these.

Illustrations of perfect solutions are unnecessary, because they occur most frequently in the study of differential equations.

Article 2.

If a differential equation of the first order is linear and rational there is one value of $\frac{dy}{dx}$ for any values of x and y and consequently one set of values for $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc. The equation accordingly represents a system of curves which do not cut one another. For if two curves of the system cut one another at a point there would be two values of $\frac{dy}{dx}$ at that point corresponding to the directions of the two curves, contrary to the suppositions of the case.

Neither can two curves of the system represented by a rational and linear differential equation of the first order have contact of any order with one another. For the condition that two curves may have a contact of the n^{th} order with one another, is, that the differential coefficient of y up to $\frac{d^ny}{dx^n}$ may be equal for the two curves at the point, but the higher differential coefficients are unequal. Since in the case of the equation considered at present, there is but one set of values for $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc., however far we may proceed, it is clear that a point moving according to the equation cannot represent a system of curves two of which have a contact of the n^{th} order with one another, however large n may be. For that would require that

$$\frac{d^{n+1}y}{dx^{n+1}},\ \frac{d^{n+2}y}{dx^{n+2}},$$

etc., will have two different values at the point, contrary to the nature of equations considered.

A rational linear differential equation of the first order accordingly represents a system of curves no two of which have a common point. A point would be moving according to the differential equation only if it described one of these curves. At any point there is but one way to go, and passage from one curve of the system to another is impossible. The complete solution therefore consists of a singly infinite system of curves, the analytical equivalent of which is an equation containing one arbitrary constant.

Illustrations are given below in figures 2, 3 and 4. The entire system of curves taken collectively may be said to be the *geometrical complete primitive*, whereas any particular curve taken by itself is a *geometrical particular solution*. Similarly the equation for the entire system with one arbitrary constant is called the *analytical complete primitive*; but when we give a particular value to the constant in that equation we get an *analytical particular solution*.

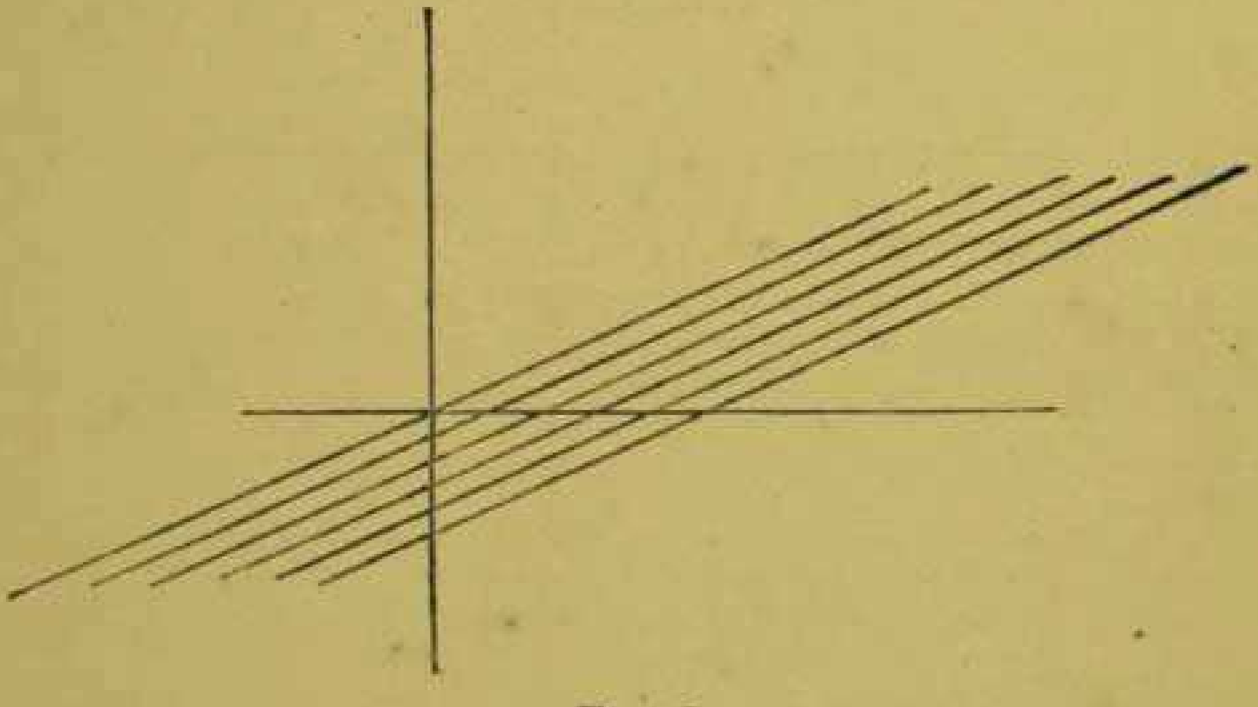

FIG. 2.

$$\frac{dy}{dx}=a$$

$$y=ax+m$$

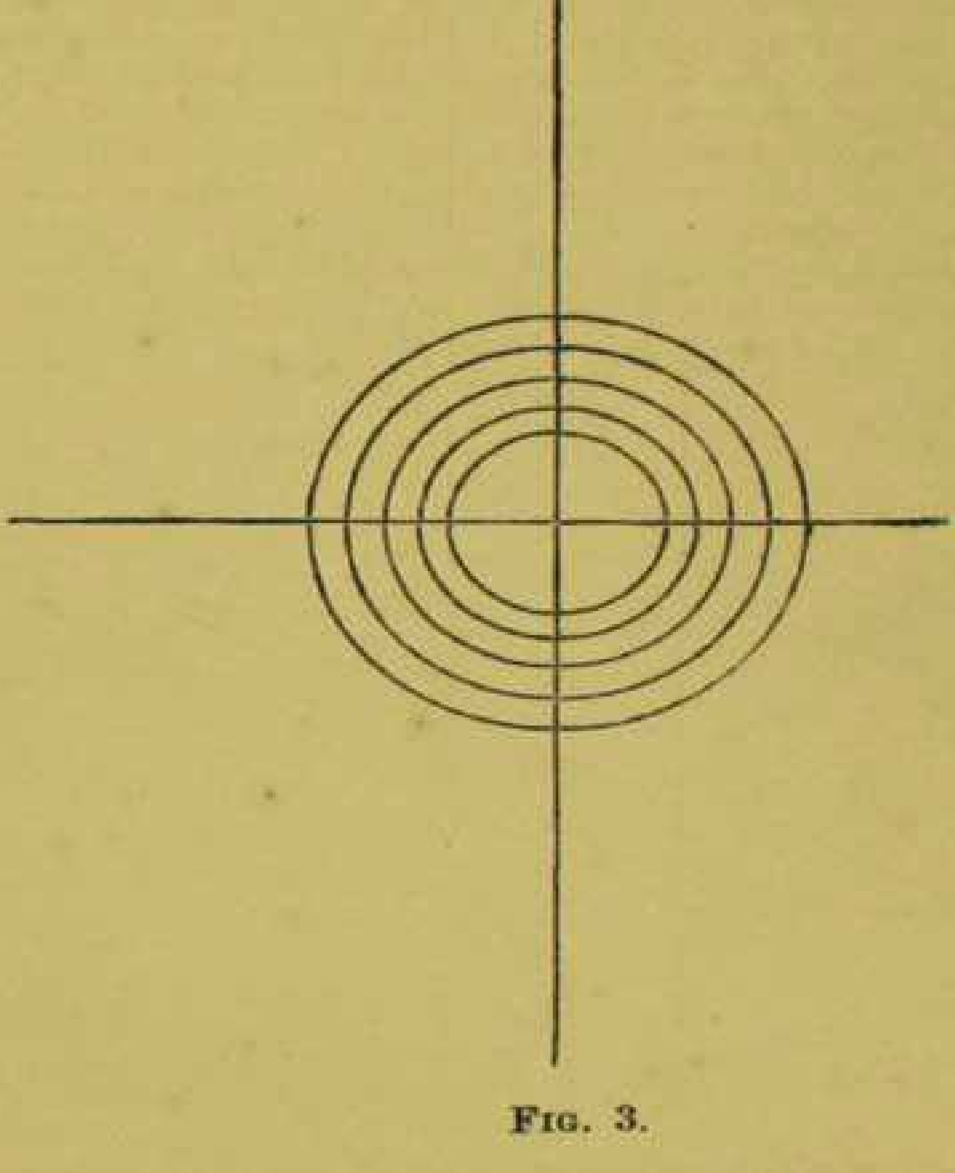

FIG. 3.

$$x + y\frac{dy}{dx} = 0$$

$$x^2 + y^2 = k^2$$

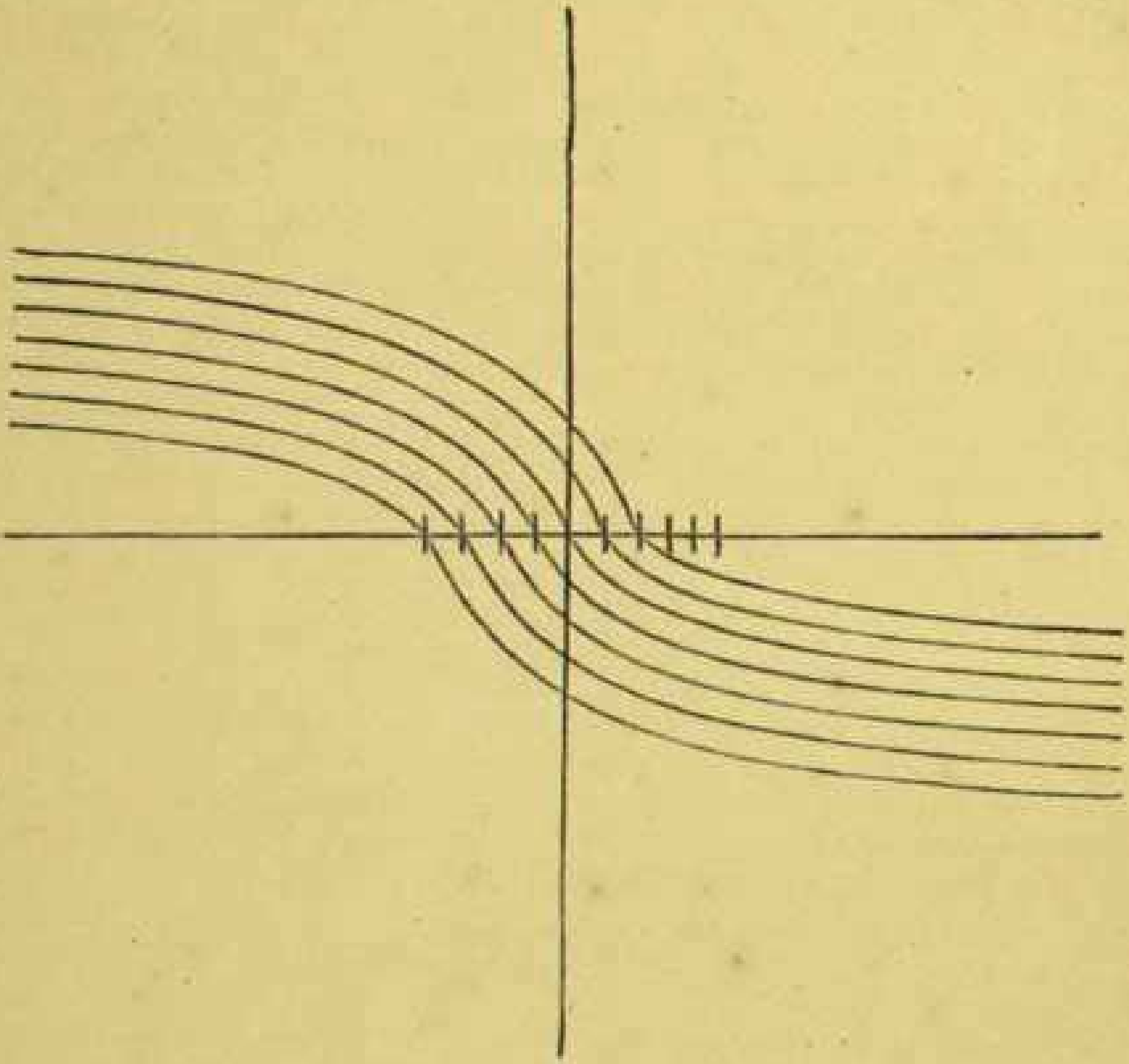

FIG. 4.

$$\frac{dy}{dx} = 3x^2$$

$$y = x^3 + c$$

Evidently it is not possible to construct any geometrical solution in this case by taking different parts of the different geometrical particular solutions of the differential equation. For, a point cannot pass from one curve of the system to another in obedience to the law of motion dictated by the differential equation. If it is on one of the curves at any instant it must always remain on that curve—it cannot pass to another except under very peculiar circumstances to be discussed hereafter (Art. 8). A linear equation of the first order has accordingly in general no other solution than those included in the complete primitive—no new geometrical solutions can be evolved by taking different parts of the different particular solutions.

Thus in any of the annexed figures 2, 3 or 4, it would be found impossible to form a continuous line by taking different parts from different curves of the system. If a point moving along one of these curves passes to the next, then that part of its path by which it passes from one curve to another is not described in accordance with the differential equation. The curve traced by the point, consisting of different parts of the two curves, is accordingly not a geometrical solution. The system of complete primitive as traced, therefore, embraces all possible solutions of the equation. In the phraseology employed in this paper hereafter there are no derived geometrical solutions of this class of differential equations.

Article 3.

We now proceed to consider the case of differential equations of the first order which are not of the first degree ; and we shall first take up such of these equations of higher degrees which are resolvable into two or more linear and rational equations.

We shall commence with the simplest example,

$$\left(\frac{dy}{dx}\right)^2 - (a+b)\,\frac{dy}{dx} + ab = 0.$$

This immediately resolves thus—

$$\left(\frac{dy}{dx} - a\right)\left(\frac{dy}{dx} - b\right) = 0 \text{ (i)}$$

In the first place it is clear that if a point moves so as always to satisfy either of the component equations

$$\frac{dy}{dx} - a = 0 \text{ (ii)}, \quad \frac{dy}{dx} - b = 0 \text{ (iii)},$$

its motion would be in accordance with the whole equation (i) as it stands. So that if we trace the curve-systems represented by (ii) and (iii), all the curves contained in them would be geometrical particular solutions of (i). Also at any point (i) gives two and only two values of $\frac{dy}{dx}$; and these correspond to the directions of the two curves of the systems (ii) and (iii) passing through that point. Thus as we see on one hand, that all solutions of (ii) and (iii) are also solutions of (i), so on the other at any point on the plane the elements of the curve-system represented by (i), are none else than the elements of the curves of the two subsidiary systems passing through that point.

Thus in a way the two curve systems represented by (ii) and (iii) may be said to constitute the geometrical complete primitive of (i), and this is true in this sense, that if a point is to move in the co-ordinate plane according to the dictates of equation (i) it must be at every instant describing some portion of a curve belonging to either of the subsidiary systems. It cannot move in a curve not included in (ii) or (iii), for then $\frac{dy}{dx}$ would not be given by equation (i). But it does not follow by any means in this case, as it followed in the last article, that if a point begins to move say in a particular curve of system (ii) it must always continue to move in that curve. For though (ii) in itself is a non-intersecting system and it is not possible for the point to pass to the contiguous curve of system (ii) without breaking the laws of motion dictated by that differential equation, all that equation (i) requires is that the point should always move *either* in a curve belonging to (ii) or in one belonging to (iii). And the moving point is free to leave at any instant the curve of system (ii) which it has been hitherto describing, and turn to the curve of system (iii), which passes through its position for the time being.

Thus in our present case the systems (ii) and (iii) are systems of straight lines represented respectively by the equations

$$y = ax + k_1 \quad (\text{ii})'$$
$$y = bx + k_2 \quad (\text{iii})'$$

where k_1 and k_2 are arbitrary constants. But that is not the only way in which we may take several rectilinear elements in the figure to arrive at a solution of (i). If we draw any line at random, only taking care that every part of it is rectilinear and coincident with a straight line of one of the systems, *e.g.*, the line (L_1) in fig. 5, it is, geometrically considered, as good a particular solution of the equation as any line of the systems (ii) or (iii). Again it may easily be seen that if instead of drawing the new line at random we draw it by systematically taking a certain length in one direction, then a certain length in the other direction, then again a certain length in the first, and so on, we can easily exhaust all the elements of the two infinite systems of straight lines by a system of sinuous lines. The truth of this will at once be apparent by examining the systems marked S_1, S_2, and S_3 in fig. 5 where, by taking any of the curves forming a system and moving it parallel to itself, we exhaust all the rectilinear elements forming (ii) and (iii).

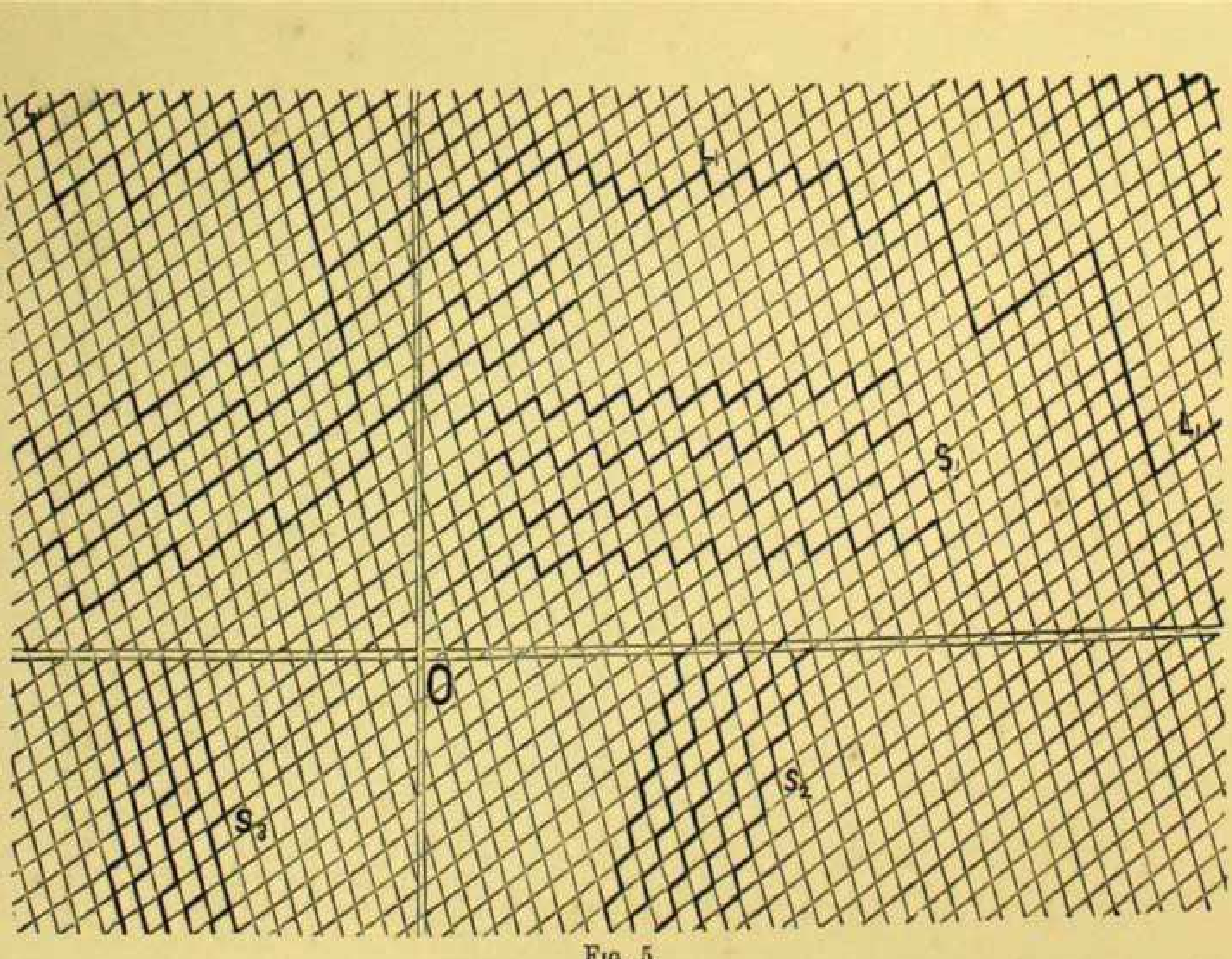

Fig. 5.

Now S_1, S_2, S_3 are as good geometrical complete primitives of (i) as the systems (ii) and (iii) taken together (which we will call S). The same elements are embraced by all of them—the order in which they are taken is only different. Yet this difference is essential and exceedingly important. For when we speak of a system of curves it is the form of an individual curve that chiefly determines the nature of the analytical equation and properties of the system. Thus in the present case if S, S_1, S_2, S_3 were all capable of analytical representation by equations we would expect these equations to be very different from one another.

Thus we see that if we have a differential equation of the first order, but not of the first degree, which is resolvable into linear equations, the geometrical complete primitives of all the component equations taken together form a geometrical complete primitive of the original differential equation. But this is not the only G.C.P. An infinite number of other G.C.P.'s can be formed by taking different parts out of the different curves forming the first geometric complete primitive. It will be convenient to call the first G.C.P. obtained by solving the component equations *the fundamental* G.C.P., whereas all others may be distinguished as *derived* G.C.P.'s of the differential equation.

The question whether systems of derived G.C.P.'s, such as S_1, S_2, S_3, etc., are expressible by functional relations between x and y involving arbitrary constants will be discussed in the next article. But it may be here stated that if they be expressible analytically they would stand in exactly the same relation to the original differential equation as the fundamental G.C.P., S. In fact the fundamental G.C.P. would correspond to the complete primitive in the equation of finite differences, whereas derived G.C.P.'s as S_1, S_2, S_3 would correspond to indirect integrals. And just as the indirect integrals of an equation of finite differences are nothing but complete primitives of the original difference equation transformed in some way, so we would expect in the case of differential equations the derived G.C.P.'s to come out as fundamental G.C.P.'s if in each case the differential equation is transformed in a suitable way before the process of integration is commenced. The interesting subject of tracing an analogy between the solutions of difference and differential equations will be taken up for fuller treatment later on.

We add a few more illustrations of the fact that a resoluble equation of the first order, but of a higher degree, has an infinite number of geometrical complete primitives. Whether these G.C.P.'s are perfect, we shall discuss in the next article. But it may be well to state here (as will be shown in the last section) that if instead of dealing with continuous curves we deal with point series, the points in a series following one another, however closely, solutions like those shown in the figure can be expressed by unctional relations between U_n and x, and are accordingly perfect.

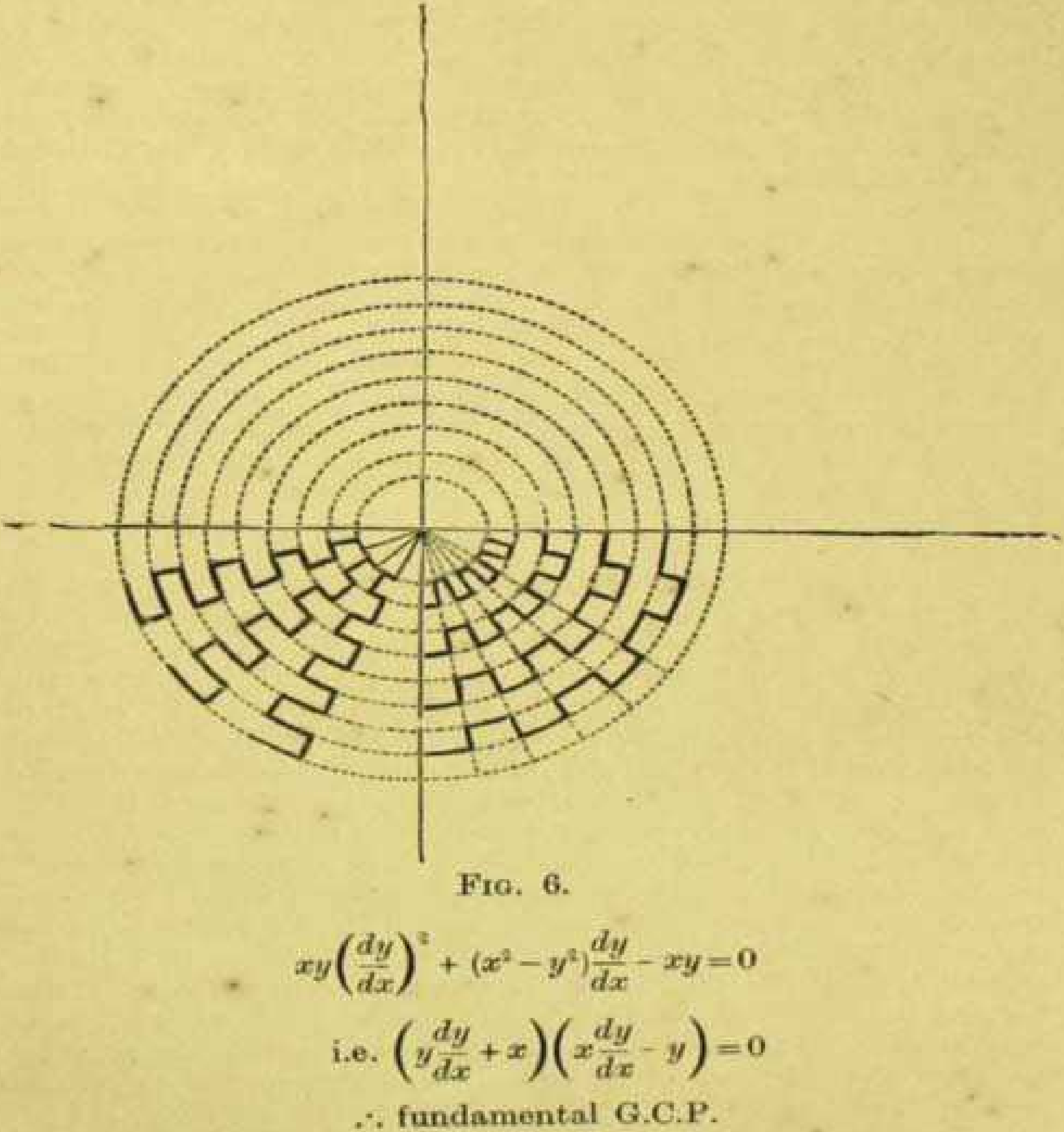

FIG. 6.

$$xy\left(\frac{dy}{dx}\right)^2 + (x^2 - y^2)\frac{dy}{dx} - xy = 0$$

$$\text{i.e. } \left(y\frac{dy}{dx} + x\right)\left(x\frac{dy}{dx} - y\right) = 0$$

$\therefore$ fundamental G.C.P.

$$\left.\begin{array}{l} x^2 + y^2 = k^2 \\ \quad y = mx \end{array}\right\} \begin{array}{l} \text{circles} \\ \text{str. lines} \end{array}$$

k and m being arbitrary constants.

Two derived G.C.P.'s are shewn in the two quadrants.

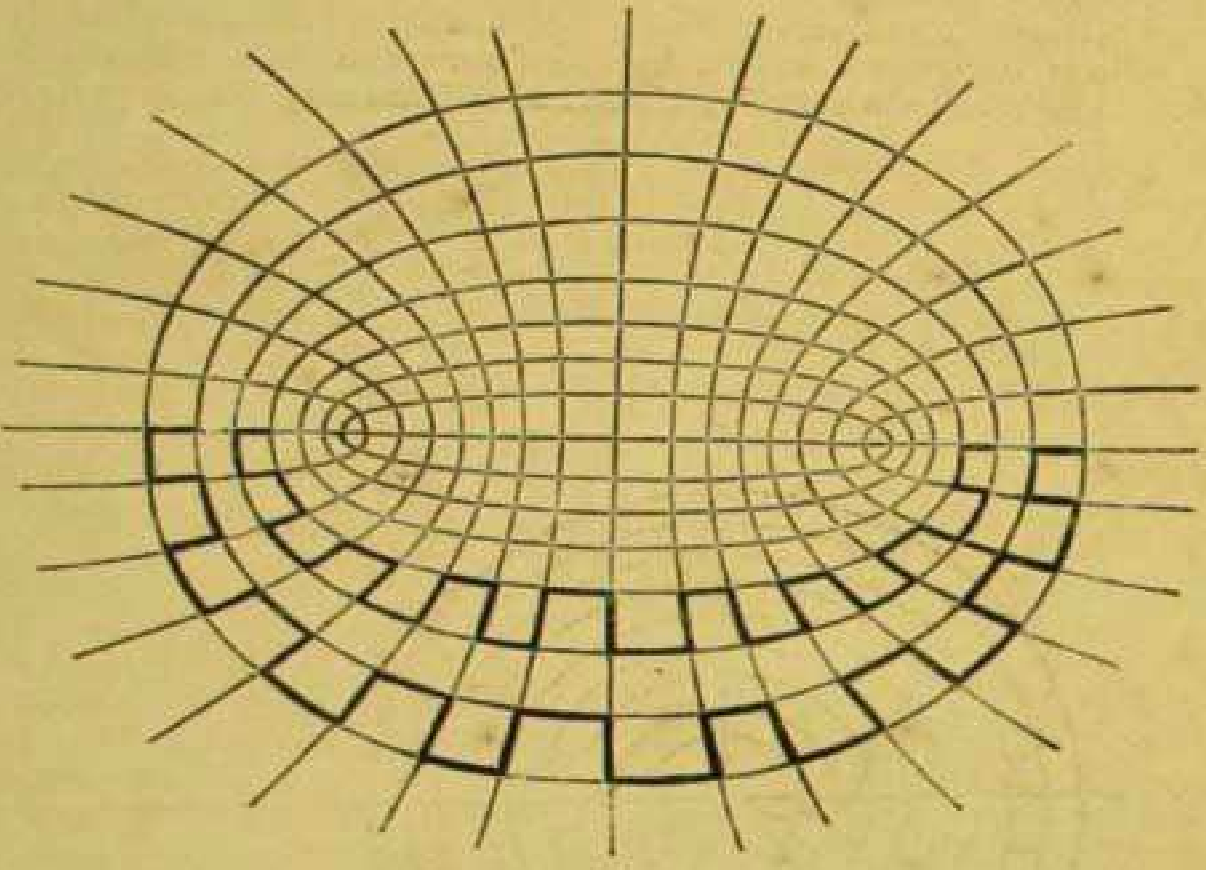

FIG. 7.

$$\frac{y^2}{b^2(b^2+\lambda^2)}\left(\frac{dy}{dx}\right)^2 + xy\left\{\frac{1}{a^2(b^2+\lambda^2)}+\frac{1}{b^2(a^2+\lambda^2)}\right\}\frac{dy}{dx}+\frac{x^2}{a^2(a^2+\lambda^2)}=0$$

$$\text{i.e.}\left\{\frac{x}{a^2}+\frac{y}{b^2}\cdot\frac{dy}{dx}\right\}\left\{\frac{x}{a^2+\lambda^2}+\frac{y}{b^2+\lambda^2}\cdot\frac{dy}{dx}\right\}$$

Fundamental G.C.P. consists of two curve systems, *viz.*, concentric conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=k$$

$$\frac{x^2}{a^2+\lambda^2}+\frac{y^2}{b^2+\lambda^2}=k^1$$

Mode of forming derived geometrical solutions illustrated.

Article 4.

In the last article we saw that a resoluble differential equation of the first order has an infinite number of geometrical complete primitives, and consequently a doubly infinite system of geometrical particular solutions. Of these we have as yet seen the analytical equivalents of only one G.C.P. (*viz.*, the fundamental one), and the infinite number of G.P.S.'s contained in it. We shall now proceed to examine whether any of the derived G.C.P.'s or derived G.P.S.'s are capable of equational representation in the ordinary system of cartesian co-ordinates, and in this connection we have slightly to touch upon the subject of analytical representation of geometrical figures.

In the first place we ask ourselves the broad and important question—if we draw any curve at random, is there necessarily an equation for it ? A little consideration will show that the answer to this question must be in the negative. An equation implies that the curve should be described according to a fixed law, and unless we had in view a law or fixed principle relating to the succession of points on the curve at the time of its description, no equation can be found for it. Nor can it be maintained that given a curve, we are given the positions of the successive points and hence also the law according to which the points follow one another. For to be given a point on the curve and another following it is not to be given the law according to which the point moves, but simply a condition which that law must satisfy. Hence to be given a finite portion of an irregularly described curve is to be given an infinite number of points following one another arbitrarily ; and this means that an infinite number of conditions is imposed upon the law according to which the moving point describes the curve. If the law is expressed as an equation, in any system of co-ordinates, every condition in the law means an independent constant. Hence whether we attempt to represent the curve by a rectangular, polar or intrinsic equation, in every case we get an infinite number of independent constants introduced, and the equation comes to nothing.

If all curves which we can draw cannot be represented by equations, it seems natural that curves, which are capable of equational representation, should have peculiar properties of their own. And one of the most important of these properties which will be useful in discussing the question of equational representation of derived G.C.P.'s of differential equations is enunciated in the following

Proposition.—If portions of arcs, however small, of two curves, representable by equations, absolutely coincide, they must coincide throughout the entire length of one of them.

It should be carefully noted that by a portion of arc of a curve is meant a continuous length of its arc containing an infinite number of points. If two curves have with one another contact of the n^{th} order, then however large n may be, the curves have only a finite number, *viz.*, $n+1$ points common; and as a finite number of points can never give a continuous length of arc, such curves cannot be said to have a portion of arc common with one another. The difference between contact and coincidence of arc is more easily perceptible from the figure given below. If two curves, A and B, meet in P and we draw PM the ordinate, take MN a small length and draw a parallel to PM through N, meeting the curves in Q, Q^1, then if the curves merely touch one another with any order of contact, QQ^1 is a small quantity of a higher order than MN, as in the case of the two lines to the right. Whereas if the curves coincide through any length, however small, at P, QQ^1 is absolutely zero, as in the case of the two lines to the left.

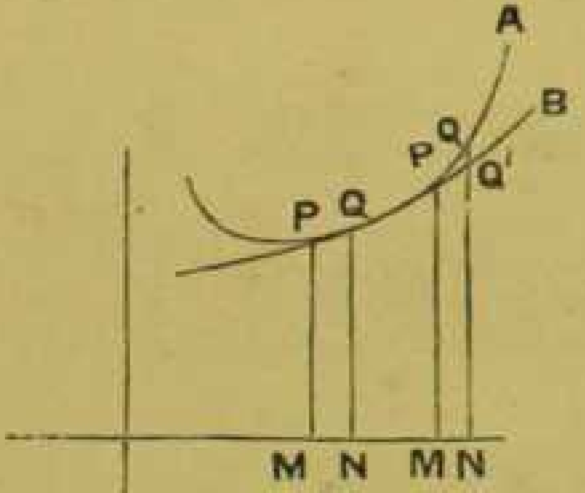

FIG. 7(A).

If possible let $y=\phi(x)$ and $y=f(x)$ be the equations of the two curves which coincide with one another from the point whose abcissa is a to the point whose abcissa is b.

Then for the ordinates of the two curves corresponding to the abcissa $a+h$, we have

$$\phi(a+h)=\phi(a)+\phi'(a)h+\frac{h^2}{\underline{|2}}\,\phi''(a)+\frac{h^3}{\underline{|3}}\,\phi'''(a)+\&c.,$$

and $$f(a+h) = f(a) + f'(a)h + \frac{h^2}{\underline{|2}} f''(a) + \frac{h^3}{\underline{|3}} f'''(a) + \&c.$$

If h is less than $b-a$, we have, since the curves coincide,

$$\{\phi(a) - f(a)\} + \{\phi'(a) - f'(a)\}h + \{\phi''(a) - f''(a)\}\frac{h^2}{\underline{|2}} + \&c. = 0$$

which for convenience we write

$$A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \&c = 0. \quad \text{(A)}$$

Now the left-hand side of (A) is an expression such that, when h is indefinitely small the different terms of it are of different order of small quantities. Also since A_0, A_1, A_2, etc., are finite quantities the sum of any number of terms in (A) cannot be equal to any other term. Thus if we take all terms from A_3h^3 upwards and add them, we get $h^3\{A_3 + A_4h + A_5h^2 + \&c., \text{ up to } \alpha\}$, which, if A_k be the largest of the quantities A_3, A_4 &c., is less than $A_kh^3\{1 + h + h^2 + \&c., \text{ up to } \alpha\}$, *i.e.*, less than A_kh^3 and is in consequence infinitely smaller than any of the preceding terms. Hence we see that the sum of a series like (A) in which terms cannot destroy each other, absolutely vanish unless each term is separately equal to zero. Hence A_0, A_1, A_2, &c., vanish separately, *i.e.*, $f(a) = \phi(a)$, $f'(a) = \phi'(a)$, $f''(a) = \phi''(a)$ &c., as far as we like.

Hence $f(a+h)$ and $\phi(a+h)$ are identical, whatever h may be, *i.e.*, the curves coincide throughout.

From this proposition we can easily pass to another which has a direct bearing on the subject of our study. It may be stated thus :—

If $\phi(x, y, a) = 0$ represents a family of curves, a being a variable parameter, then no equation can be obtained for a curve which contains as its parts portions of arc, however small, from an infinite number of curves of the system.

For if possible let $f(x,y) = 0$ be a curve which has as its parts portions of arcs of an infinite number of curves of the system $\phi(x, y, a)$. Then by the previous proposition $f(x,y)$ coincides with each of the infinite number of different curves of the system. But this is impossible. Hence there can be no such equation as $f(x, y) = 0$ for the curve in question.

It should be carefully observed that the last theorem proves to be analytically unrepresentable such derived curves of the system as contain *portions of arcs* from an infinite number of curves of the system. But if we can form a continuous curve by taking not continuous lengths of arcs but a finite number (2 or 3 or n)[1] of points on each, that curve may be expressible by an equation. Such a curve may be one which touches every curve of the system or has contact of a higher order with every one of them. In fact the envelope of the system is a curve of this description, and we know it to be analytically expressible.

Article 5.

We are now in a position to examine whether any of the infinite number of derived G.C.P.'s or G.P.S.'s of the class of differential equations discussed in Art. 3 are perfect solutions, *i.e.*, solutions geometrically as well as analytically. With this view it is only necessary to examine whether there is any among the curves which can be represented by an equation.

In the first place we at once discard such curves as are shown in figs. 5 and 6 and 7, for though faultless as geometrical solutions they consist of portions of arcs of an infinite number of curves of a system, and are accordingly, from what has been proved in Art. 4, incapable of analytical representation.

The only thing then which we have to examine is whether a G.S. can be formed by taking a finite number of points on each curve. This must obviously be such a curve that every point on it would be a point on one of the curves of the system, and the direction at that point of it must be along the tangent to that curve of the system, and yet no part of its arc shall coincide with a part of that curve of the system. The envelopes of the system are clearly the only curves which satisfy all these conditions in relation to the system.

But if a system consists of n sub-systems, each of which is non-intersecting, it is clear that the compound system can have no envelope. For the envelope of the whole system must also envelope any of the non-intersecting sub-systems, and it is impossible to conceive of an envelope in the case of a family of curves the members of which do not cut one another; for an envelope is defined as the locus of the ultimate points of intersection.

---

[1] A curve formed by taking one point on each of the curves constituting the fundamental geometrical complete primitive will not necessarily be a derived particular solution of the differential equation; for its direction will not at any point coincide with the direction of the fundamental particular solution through that point. In fact any arbitrary curve traced in the plane may be supposed to be made up of one point from each fundamental particular solution. Under peculiar circumstances, however, such curves are not only derived geometrical particular solutions but are such as are perfect, *i.e.*, analytically expressible.

In one sense, however, an envelope *can* be formed in such a case. For if we start from any point A, go up to the next point B, along a line of system (i), come down to c, the next point by a line of system (iii), and so on, we get a curve which is made up of two consecutive points taken from each curve of the system, and is therefore in one sense an envelope of the system. But this curve belongs to a class of curves which we cannot study analytically —perhaps not even geometrically. For it is a sinuous curve, the direction of the curve abruptly changing at every successive point ; and hence though it does not at all differ in appearance to the eye from the nodal locus passing through AC . . . it is altogether a different curve. Indeed the two curves, *viz.*, the envelope and the nodal line, are at every point indefinitely close to one another, but the directions of the two curves are in no place identical. Thus if we are considering properties which involve only the values of co-ordinates of points on a curve, the equation of the nodal line can be taken to be the equation of the sinuous envelope. But the equation will belong to the latter locus only as a relation between the abcissa and ordinate of a point on it. The ordinary process of analytical geometry and the Differential and Integral Calculus for deducing the directional properties, and those relating to curvature, etc., of a curve from its equation, will not at all be applicable in this case. This will be more clearly apparent from the following consideration relating to the ordinary study of curves.

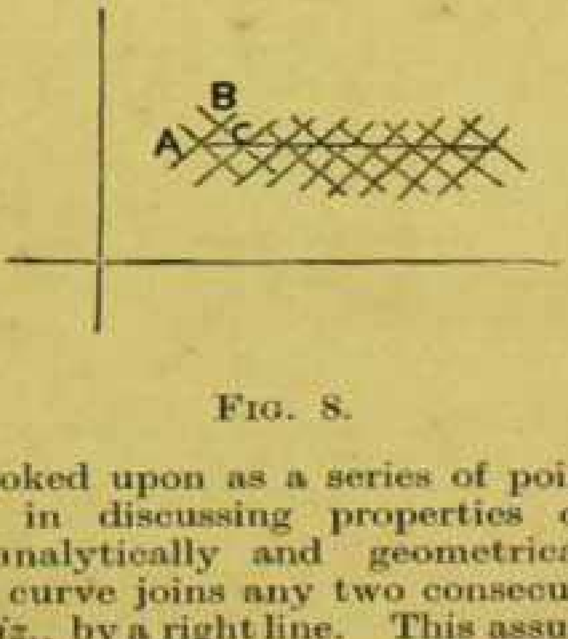

FIG. 8.

A curve is looked upon as a series of points arranged consecutively. Now in discussing properties of tangency and curvature both analytically and geometrically, it is always assumed that the curve joins any two consecutive points in the most direct way, *viz.*, by a right line. This assumption must be at the root of all studies relating to curves. For if we remember that we may join two given points in an infinite number of ways, how unimaginably vast must be the number of curves which will connect an infinite number of given consecutive points! And we would not know if a locus which appears to the eye as the right line XY is in reality the locus which it appears to be or the limit of a sinuous curve like A or B or any other. With such uncertainty attaching to the real nature of a curve the study of curves becomes impossible from a geometrical point of view; and much more would it be impossible to study such curves analytically, for we cannot imagine a function of x and y which is perpetually discontinuous for all values of the variable, and such is $\frac{dy}{dx}$ required to be in the case of such sinuous curves under consideration.

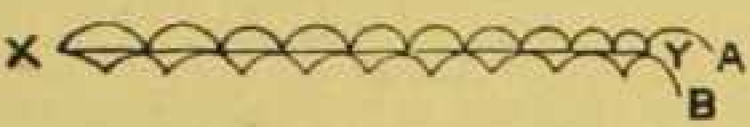

Fig. 9.

If then by a curve we mean a curve in the ordinary geometrical and analytical sense of the term we may assert what we have said already, that there cannot be a *curve* touching all the members of a system of curves consisting of two or more non-intersecting sub-systems.

Thus it follows that all such derived geometrical solutions as can be formed in the case of resoluble differential equations of the first order are incapable of analytical representation, and consequently the fundamental solutions are the only ones which are perfect solutions from every point of view. Such an equation accordingly has one perfect C.P. and an infinite number of particular solutions constituting it.

Article 6.

We shall now proceed to consider the case of those differential equations of the first order which are not resoluble into two or more rational and linear equations such as were considered in Art. 2. We shall find it convenient to study this class of equations by comparing them with the class of resolvable equations considered in Arts. 3-5 and examining how far the property of being resolvable affects the nature of geometrical and analytical solutions. We shall find as we proceed that so far as regards the *method* of solution the analogy is complete and the process adopted for solving resolvable equations is perfectly applicable to irresolvable equations as well. As regards the geometrical representation we shall find that the analogy *may* be pushed, and we *may* look upon the geometrical loci represented by irresolvable equations as several systems of non-intersecting curves, as in the case of resolvable equations ; though it must be noticed that this is neither the simplest nor the most convenient view of the thing. But it is as regards the analytical representation of derived geometrical solutions that the difference will be striking. For whereas we found all derived G.P.S.'s of resolvable equations unrepresentable, one derived G.P.S. of an irresolvable equation may frequently be represented by an equation.

In the first place we turn to the method of solution. The method of solving resolvable equations is to solve the component linear and rational equations and multiply up the solutions after replacing the arbitrary constant in each by the same symbol. Exactly the same method is applicable to equations irresolvable into real linear and rational components. For if we resolve such an equation into factors, not minding the irrationals or imaginaries that occur, and then deal with these factors exactly as we do with linear real and rational factors, we get the solution of the equation, though of course in many cases there may exist a simpler and shorter method of arriving at it. We shall illustrate this fact with one or two examples.

Take the equation $xp^2-2yp+4x=0$ (i). Rearranging the terms thus $y=\frac{x}{2}p+\frac{2}{p}x$ (ii) we observe that the equation is in Clairaut's form. Accordingly differentiating,

$$\frac{dy}{dx}=p=\tfrac{1}{2}p+\frac{x}{2}\cdot\frac{dp}{dx}+2\frac{p-x\frac{dp}{dx}}{p^2}$$

$$\text{or }\left(\frac{p}{2}-\frac{2}{p}\right)=x\frac{dp}{dx}\left\{\tfrac{1}{2}-\frac{2}{p^2}\right\}.$$

Hence neglecting the factor p^2-4, which does not lead to a complete primitive, we have

$$\frac{dp}{p}=\frac{dx}{x}\text{ or }p=cx\text{ (iii).}$$

Substituting for p from (iii) in (ii) we get the solution

$$x.\,c^2x^2-2\,y.cx+4x=0,$$

or neglecting $x=0$ which is not a solution

$$x^2=\frac{2}{c}\left(y-\frac{2}{c}\right)$$

$$=c\,(y-c)\quad\text{(iv)}$$

since c is an arbitrary constant.

This is the complete primitive. Now if we had resolved (i) into two component equations, and solved the irrational component equations separately, then the solutions obtained when multiplied together, after the arbitrary constants in both had been replaced by the same symbol, would lead to (iv) exactly

Thus $\qquad xp^2-2yp+4x=0$

Solving $\qquad xp=y\pm\sqrt{y^2-4x^2}.$

Put $\qquad y=vx$

Then $\qquad x\left(v+x\dfrac{dv}{dx}\right)=vx\pm x\sqrt{v^2-4}$

$$v+x\frac{dv}{dx}=v\pm\sqrt{v^2-4}$$

$$\frac{dv}{\pm\sqrt{v^2-4}}=\frac{dx}{x}.\quad\text{(v)}$$

Taking the positive sign and integrating

$$c+\log x=\int\frac{dv}{\sqrt{v^2-4}}=\log\left(v+\sqrt{v^2-4}\right)$$

$$\text{or } cx=\frac{y}{x}+\sqrt{\frac{y^2}{x^2}-4}.$$

$$\text{i.e., } cx^2=y+\sqrt{y^2-4x^2}$$

which may be put in the form

$$c=y-\sqrt{y^2-4x^2}$$

since c is an arbitrary constant.

If we had taken the negative sign of the radical in (v) we would get $c'=y+\sqrt{y^2-4x^2}$.

BCU 1717

The solution then ought to be

$$(y-c+\sqrt{y^2-4x^2})(y-c-\sqrt{y^2-4x^2})=0$$

$$\text{i.e. } (y-c)^2-y^2+4x^2=0$$

$$-2cy+c^2+4x^2=0$$

$$\text{i.e., } x^2=\frac{c}{2}\left\{y-\frac{c}{2}\right\}$$

$$=c\,(y-c)$$

as we got in the previous method of solution.

To take another example—

$$(x^2-a^2)\,p^2-2xyp-x^2=0 \text{ (vi).}$$

This is also in Clairaut's form when put in the form—

$$y=\frac{(x^2-a^2)\,p^2}{2xp}-\frac{x^2}{2xp}$$

$$=\frac{(x^2-a^2)\,p}{2x}-\frac{x}{2p}.$$

Differentiating,

$$p=\frac{x}{2}\cdot\frac{dp}{dx}+\frac{p}{2}-\frac{a^2}{2}\left\{\frac{x\frac{dp}{dx}-p}{x^2}\right\}-\tfrac{1}{2}\cdot\frac{p-x\frac{dp}{dx}}{p^2}$$

$$\therefore \frac{p}{2}-\frac{a^2p}{2x^2}+\frac{1}{2p}=\frac{dp}{dx}\left\{\frac{u}{2}-\frac{a^2}{2x}+\frac{x}{2p^2}\right\}.$$

Hence neglecting the factor $x^2p^2-a^2p^2+x^2=0$ which cannot, when combined with (vi), give rise to a complete primitive since it involves no arbitrary constant, we get $p=cx$ (vii): substituting which in (vi) we get, after easy transformations, the complete primitive in the form $x^2-a^2-2cy-c^2=0$.

We shall shew in this equation also that a method of solution analogous to resolvable equation will apply.

Solving then equation (vi) as a quadratic in p we get

$$p=\frac{xy\pm\sqrt{x^2y^2+x^2(x^2-a^2)}}{x^2-a^2}$$

$$=\frac{x}{x^2-a^2}\,(y\pm\sqrt{x^2+y^2-a^2})$$

Gcs 3668

$$dy = \frac{x\,dx}{x^2 - a^2}\left(y \pm \sqrt{x^2 + y^2 - a^2}\right).$$

Let $x^2 - a^2 = x'^2$ so that $x dx = x'\,dx'$

$$\therefore\ dy = \frac{x'\,dx'}{x'^2}\left(y \pm \sqrt{x'^2 + y^2}\right)$$

$$\therefore\ x'dy = dx'\left(y \pm \sqrt{x'^2 + y^2}\right).$$

Put $y = vx'$.

$$\text{Then } x'\left(v + x'\frac{dv}{dx'}\right) = x'\left(v \pm \sqrt{v^2 + 1}\right)$$

$$\frac{dx'}{x'} = \pm\frac{dv}{\sqrt{v^2 + 1}}.$$

Now taking the upper sign,

$$cx' = v + \sqrt{v^2 + 1}$$

$$= \frac{y}{x'} + \sqrt{\frac{y^2}{x'^2} + 1}$$

$$\therefore\ c(a^2 - x^2) = y + \sqrt{y^2 + x^2 - a^2}$$

$$c = y - \sqrt{x^2 + y^2 - a^2}$$

since c is arbitrary.

With the other sign we get

$$c' = y + \sqrt{x^2 + y^2 - a^2}.$$

Hence the solution is

$$(y - c)^2 - (x^2 + y^2 - a^2) = 0$$
$$\text{i.e., } c^2 - 2cy - x^2 + a^2 = 0,$$

which agrees with the solution obtained by the first method.

The theorem which we have attempted to illustrate above may be stated slightly differently as follows :—

If there be a differential equation of the n^{th} degree and first order $p^n + \Phi_1(x, y)\,p^{n-1} + \Phi_2(x, y)\,p^{n-2} +$ &c. $= 0$, and its solution be, as usual, represented by

$$c^n + F_1(x, y)\,c^{n-1} + F_2(xy)c^{n-2} + \&c. = 0$$

then if the differential equation be put in the form

$$\{p - \phi_1(x, y)\}\,\{p - \phi_2(x, y)\}\,\ldots\ldots\,\{p - \phi_n(xy)\}$$

and the complete primitive in the form

$$\{c-f_1(x,\ y)\}\ \{c-f_2(x,\ y)\}\ .\ .\ .\ \{c-f_k(x,\ y)\}$$

where ϕ_1, ϕ_2, etc., and f_1, f_2, etc., are any functions rational or irrational, real or imaginary of x and y, then each of the equations

$$c-f_1(x,\ y)=0,\quad c-f_2(x,y)=0\ \&c.$$

are the complete primitives of the corresponding equations amongst $p-\phi_1(xy)=0$, $p-\phi_2(x,\ y)=0$ &c.

We have illustrated the above theorem for equations of the the second degree; for we have exemplified that the complete primitive of such an equation is obtained by multiplying up the complete primitives of the two component equations. Indeed, the solution of a quadratic being a simple affair, this method of splitting up the differential equation into two or more equations, even if they are irrational or imaginary, is frequently adopted in practice.

In many cases this is the only method available for the solution of the equation. For example take the equation

$$p^2y^2\cos^2 a-2pxy\sin^2 a+y^2-x^2\sin^2 a=0\ \text{(i)}.$$

To find the complete primitive of this equation we have to solve it as a quadratic in p, and it will then be found that the two component equations, though irrational, come under the head of homogeneous equations and are easily soluble. But the equation (i) as a whole defies solution by any other method.

$$\text{Thus}\quad p=\frac{xy\sin^2 a\pm\sqrt{x^2y^2\sin^4 a-y^2\cos^2 a\ (y^2-x^2\sin^2 a)}}{y^2\cos^2 a}$$

$$=\frac{x\sin^2 a\pm\sqrt{x^2\sin^2 a-y^2\cos^2 a}}{y\cos^2 a}\qquad\text{(B) \& (C)}$$

Let $y=vx$. Then

$$\frac{dy}{dx}=v+x\frac{dv}{dx}=\frac{x\sin^2 a+x\sqrt{\sin^2 a-v^2\cos^2 a}}{vx\cos^2 a}$$

$$\therefore x\frac{dv}{dx}=\frac{\sin^2 a-v^2\cos^2 a\pm\sqrt{\sin^2 a-v^2\cos^2 a}}{v\cos^2 a}$$

$$\therefore\frac{dx}{x}=\frac{vdv\cos^2 a}{\sqrt{R}\{\sqrt{R}\pm1\}}\quad\text{(A) where } R=\sin^2 a-v^2\cos^2 a$$

Taking the upper sign we get by integration

$$\log x=\cos^2 a\int\frac{vdv}{\sqrt{R}\,(\sqrt{R}+1)}+C.$$

Let $\sin^2\alpha - v^2 \cos^2\alpha = k^2 \cos^2\alpha$

i.e., $\tan^2\alpha - v^2 = k^2$ or $vdv = -kdk$

$$\log x = -\cos^2\alpha \int \frac{kdk}{k\cos\alpha\,(k\cos\alpha + 1)} + C. = -\int \frac{dk\cos\alpha}{k\cos\alpha + 1} + C$$

$$= -\log(k\cos\alpha + 1) + C \qquad \therefore x = \frac{C}{k\cos\alpha + 1}$$

$$\text{i e., } x = \frac{C}{\sqrt{\tan^2\alpha - v^2}.\cos\alpha + 1} = \frac{C}{\sqrt{\sin^2\alpha - v^2\cos^2\alpha} + 1}$$

$$= \frac{Cx}{\sqrt{x^2\sin^2\alpha - y^2\cos^2\alpha} + x}$$

$$\text{i.e., } \sqrt{x^2\sin^2\alpha - y^2\cos^2\alpha} + x - C = 0.$$

Similarly if we take the lower sign in (A) we would get

$$-\sqrt{x^2\sin^2\alpha - y^2\cos^2\alpha} + x - c = 0.$$

Multiplying up the solutions of the component equations we get

$$(x-c)^2 - (x^2\sin^2\alpha - y^2\cos^2\alpha) = 0$$

$$\text{i.e., } x^2 - 2cx + c^2 - x^2\sin^2\alpha + y^2\cos^2\alpha = 0$$

$$\text{i.e., } x^2 + y^2 - 2c\sec^2\alpha x + c^2\sec^2\alpha = 0.$$

Putting a new constant k for $c\sec^2\alpha$ we get

$$x^2 + y^2 - 2kx + k^2\cos^2\alpha = 0,$$

which is the complete primitive of the original equation in its simplest form.

But in the case of equations of the third and higher degrees the algebraical solutions of which are long and cumbrous, the truth of the theorem is less apparent as the transformations required for verifying its truth are in some cases tedious and difficult. We shall accordingly give one or two illustrations of the theorem referring to equations of higher degrees, and then give a general proof of the theorem.

Take the equation of the third degree—

$$p^3 - 4\,xyp + 8y^2 = 0.$$

If we solve it in the ordinary way we get the complete primitive $y = c(x-c)^2$.

If we try to solve the differential equation as a cubic in p, then adopting the usual notation

$$3\ H = -4\ xy \ \therefore\ H = -\tfrac{4}{3}\ xy \text{ and } g = 8y^3$$

$$\therefore \text{ if } p = \sqrt[3]{A} + \sqrt[3]{B}$$

$$A = \tfrac{1}{2}\{-8\,y^3 + \sqrt{64\,y^4 - 4(\tfrac{4}{3}\,xy)^3}\} = 4y\,\{-y + \sqrt{y^2 - \tfrac{4}{27}\,x^3y}\}$$

$$B = \tfrac{1}{2}\{-8\,y^3 - \sqrt{64\,y^4 - 4(\tfrac{4}{3}xy)^3}\} = 4\,y\,\{-y - \sqrt{y^2 - \tfrac{4}{27}\,x^3y}\}$$

$$\therefore\ p = 4^{\frac{1}{3}}\,y^{\frac{1}{3}}\{-y + \sqrt{y^2 - \tfrac{4}{27}x^3y}\}^{\frac{1}{3}} + 4^{\frac{1}{3}}\,y^{\frac{1}{3}}\ \times$$

$$\{-y - \sqrt{y^2 - \tfrac{4}{27}\,x^3y}\}^{\frac{1}{3}}.$$

Again if we put the complete primitive as a cubic in c it stands

$$c^3 - 2\,c^2x + cx^2 - y = 0.$$

Again adopting the usual notation we find in this case

$$H = \frac{x^2}{3} - \frac{4x^2}{9} = -\frac{x^2}{9}$$

$$G = -y + 3.\ \frac{2x}{3}.\ \frac{x^2}{3} - 2\,\frac{8x^3}{27} = -y + \frac{2}{27}x^3.$$

$$\text{Hence if } z = c - \frac{2x}{3} = \sqrt[3]{A +}\ \sqrt[3]{B}$$

$$A = \tfrac{1}{2}\,\{y - \tfrac{2}{27}\,x^3 + \sqrt{y^2 - \tfrac{4}{27}x^3y}\}$$

$$B = \tfrac{1}{2}\,\{y - \tfrac{2}{27}x^3 - \sqrt{y^2 - \tfrac{4}{27}x^3y}\}$$

$$\text{and } c = \frac{2x}{3} + \frac{1}{2^{\frac{1}{3}}}\left\{\,y - \tfrac{2}{27}\,x^3 + \sqrt{y^2 - \tfrac{4}{27}x^3y}\right\}^{\frac{1}{3}}$$

$$+ \frac{1}{2^{\frac{1}{3}}}\left\{y - \tfrac{2}{27}\ x^3 - \sqrt{y^2 + \tfrac{4}{27}x^3y}\right\}^{\frac{1}{3}} \text{(ii).}$$

The question now is whether (ii) is a solution of (i), *i.e.*, whether the value of $\frac{dy}{dx}$ obtained by differentiating (ii) is identical with that given by (i).

Now if we proceed to differentiate (ii) immediately as it stands, the expression for $\frac{dy}{dx}$ will become so long and cumbrous that it would be exceedingly troublesome, if at all possible, to identify it with the value given by (i). Hence it is useful to perform some transformations prior to differentiation which will make the operations simpler.

If we notice that

$$\left(-y+\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^2=2y^2-\tfrac{4}{27}x^3y-2y\sqrt{y^2-\tfrac{4}{27}x^3y}$$

$$=2y\left\{y-\tfrac{2}{27}x^3-\sqrt{y^2-\tfrac{4}{27}x^3y}\right\}$$

we see that,

$$c=\frac{2x}{3}+\frac{1}{2^{\frac{1}{3}}(2y)^{\frac{1}{3}}}\left(-y+\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^{\frac{2}{3}}+\frac{1}{2^{\frac{1}{3}}(2y)^{\frac{1}{3}}}$$

$$\left(-y-\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^{\frac{2}{3}}$$

$$\text{or } c'=\frac{2^{\frac{2}{3}}x}{3}+y^{-\frac{1}{3}}\left\{\left(-y+\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^{\frac{2}{3}}\right.$$

$$\left.+\left(-y-\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^{\frac{2}{3}}\right\}$$

$$=y^{-\frac{1}{3}}\left\{\left(-y+\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^{\frac{1}{3}}+\left(-y-\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^{\frac{1}{3}}\right\}^2$$

$$\therefore c''=y^{-\frac{1}{6}}\left\{\left(-y+\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^{\frac{1}{3}}+\left(-y-\sqrt{y^2-\tfrac{4}{27}x^3y}\right)^{\frac{1}{3}}\right\}$$

$$=\left\{-y^{\frac{1}{2}}+\sqrt{y-\tfrac{4}{27}x^3}\right\}^{\frac{1}{3}}+\left\{-y^{\frac{1}{2}}-\sqrt{y-\tfrac{4}{27}x^3}\right\}^{\frac{1}{3}}$$

$$=R^{\frac{1}{3}}+R'^{\frac{1}{3}}\text{ say}\quad\text{(iii).}$$

Now if we differentiate the equation (ii) thus transformed we easily get the value of $\frac{dy}{dx}$ given by (i).

Thus differentiating (iii) we get

$$0=\tfrac{1}{3}R^{-\frac{2}{3}}\left\{-\tfrac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}+\tfrac{1}{2}\gamma^{-\frac{1}{2}}\left(\frac{dy}{dx}-\tfrac{4}{9}x^2\right)\right\}$$

$$+\tfrac{1}{3}R'^{-\frac{2}{3}}\left\{-\tfrac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}-\tfrac{1}{2}\gamma^{-\frac{1}{2}}\left(\frac{dy}{dx}-\tfrac{4}{9}x^2\right)\right\}$$

$$\therefore\frac{dy}{dx}\left\{-y^{-\frac{1}{2}}\left(R^{-\frac{2}{3}}+R'^{-\frac{2}{3}}\right)+\gamma^{-\frac{1}{2}}\left(R^{-\frac{2}{3}}-R'^{-\frac{2}{3}}\right)\right\}$$

$$=\frac{4}{9}x^2\gamma^{-\frac{1}{2}}\left(R^{-\frac{2}{3}}-R'^{-\frac{2}{3}}\right)$$

Multiplying both sides by $y^{\frac{1}{2}}\gamma^{\frac{1}{2}}R^{\frac{2}{3}}y^{\frac{1}{3}}$

$$\therefore \frac{dy}{dx}=\frac{\frac{4}{3}x^2\,(R^{\frac{1}{3}}-R'^{\frac{1}{3}})y^{\frac{1}{3}}}{y^{\frac{1}{2}}(R^{\frac{1}{3}}-R'^{\frac{1}{3}})+(R^{\frac{1}{3}}+R'^{\frac{1}{3}})\gamma^{\frac{1}{2}}}$$

$$=\frac{\frac{4}{3}x^2(R^{\frac{2}{3}}-R'^{\frac{2}{3}})y^{\frac{1}{2}}}{R^{\frac{2}{3}}(y^{\frac{1}{2}}+\gamma^{\frac{1}{2}})-R'^{\frac{2}{3}}(y^{\frac{1}{2}}-\gamma^{\frac{1}{2}})}$$

$$=\frac{\frac{4}{3}x^2\,(R^{\frac{2}{3}}-R'^{\frac{2}{3}})y}{-R^{\frac{2}{3}}.R'+R'^{\frac{2}{3}}R}$$

$$=\frac{\frac{4}{3}y^{\frac{1}{3}}x^2(R^{\frac{1}{3}}+R'^{\frac{1}{3}})}{R^{\frac{1}{3}}R'^{\frac{1}{3}}}$$

$$=\frac{4}{4^{\frac{2}{3}}}\,y^{\frac{1}{3}}(R^{\frac{1}{3}}+R'^{\frac{1}{3}})$$

$$=4^{\frac{1}{3}}y^{\frac{1}{3}}(-y+\sqrt{y^2-\tfrac{4}{27}x^3y})^{\frac{1}{3}}+4^{\frac{1}{3}}y^{\frac{1}{3}}(-y-\sqrt{y^2-\tfrac{4}{27}x^3y})$$

the same value as is given by differential equation (i).

It is evident that if instead of starting with (ii) *viz.*,

$$c=\frac{2x}{3}+\frac{1}{2^{\frac{1}{3}}}\left\{y-\tfrac{2}{27}x^3+\sqrt{y^2-\tfrac{4}{27}x^3y}\right\}^{\frac{1}{3}}$$

$$+\frac{1}{2^{\frac{1}{3}}}\left\{y-\tfrac{2}{27}x^3-\sqrt{y^2-\tfrac{4}{27}x^3y}\right\}^{\frac{1}{3}}$$

we started with another component equation of the cubic $y=c\,(x-c)^2$, e.g.,

$$c=\frac{2x}{3}+\frac{w}{2^{\frac{1}{3}}}\left\{y-\tfrac{2}{27}x^3+\sqrt{y^2-\tfrac{4}{27}x^3y}\right\}^{\frac{1}{3}}$$

$$+\frac{w^2}{2^{\frac{1}{3}}}\left\{y-\tfrac{2}{27}x^3+\sqrt{y^2-\tfrac{4}{27}x^3y}\right\}^{\frac{1}{3}}$$

we would arrive at a value of $\frac{dy}{dx}$ *viz.*—

$$4^{\frac{1}{3}}\,y^{\frac{1}{3}}(-y+\sqrt{y^2-\tfrac{4}{27}x^3y})+w.\,4^{\frac{1}{3}}\,y^{\frac{1}{3}}\;(-y-\sqrt{y^2-\tfrac{4}{27}x^3y}$$

which is a second value of $\frac{dy}{dx}$ given by original differential equation of the third degree, *viz.*, $p^3-4xyp+8y^2=0$.

Similarly the third component of the cubic complete primitive would lead to the third value of $\frac{dy}{dx}$ given by the original differential equation.

Thus the general theorem enunciated before is verified for the particular cubic equation chosen.

We shall now proceed to give a general proof of the theorem.

By hypothesis $\{c-f_1(xy)\}\{c-f_2(xy)\}\ldots\{c-f_4(xy)\}=0$

is an equation such that if it is differentiated with respect to x and the c eliminant formed, the result is

$$\{p-\phi_1(xy)\}\{p-\phi_2(xy)\}\ldots\{p-\phi_4(xy)\}.$$

We have to show that if $c-f_1(xy)=0$ is differentiated with respect to x and the c eliminant formed, the result will be $p-\phi_1(xy)=0$, and so for all other factors.

We are given the equation

$$(c-f_1)(c-f_2)\ldots(c-f_4)=0 \quad \text{(i)}$$

and we are given that if this relation subsists between x and y $\frac{dy}{dx}$ is given by the equation

$$\{p-\phi_1(xy)\}\{p-\phi_2(xy)\}\ldots\{p-\phi_4(xy)\}=0 \quad \text{(ii).}$$

Now it is clear that if (i) is satisfied by c being equal to $f_1(x,y)$, $\frac{dy}{dx}$ is given by $\frac{df_1}{dx}+\frac{df_1}{dy}\cdot\frac{dy}{dx}=0$; if (i) is satisfied by c being equal to $f_2(xy)$, $\frac{dy}{dx}$ is given by $\frac{df_2}{dx}+\frac{df_2}{dy}\cdot\frac{dy}{dx}=0$, and so on.

Hence since (i) is satisfied by c being equal to either f_1 or f_2 or f_3, etc., $\frac{dy}{dx}$ must be given by one of the equations

$$\frac{df_1}{dx}+\frac{df_1}{dy}\cdot\frac{dy}{dx}=0$$

$$\frac{df_2}{dx}+\frac{df_2}{dy}\cdot\frac{dy}{dx}=0$$

and so on, *i.e.*, by the equation

$$\left(\frac{dy}{dx}+\frac{\frac{df_1}{dx}}{\frac{df_1}{dy}}\right)\left(\frac{dy}{dx}+\frac{\frac{df_2}{dx}}{\frac{df_2}{dy}}\right)\cdots\left(\frac{dy}{dx}+\frac{\frac{df_n}{dx}}{\frac{df_n}{dy}}\right)=0.$$

Hence this equation must be identical with

$$\{p-\phi_1(xy)\}\{p-\phi_2(xy)\}\ .\ .\ .\ \{p-\phi_n(xy)\}=0,$$

and hence we get that the differentiation of $f_1=c$, $f_2=c$, etc., gives the respective differential equations

$$p-\phi_1(xy)=0,\ p-\phi_2(xy)=0,\ \text{etc.},$$

which is the thing sought to be proved.

Article 7.

We now proceed to consider the question of geometrical interpretation of irresoluble differential equations of the first order and their complete primitives. We have seen that a resolvable equation of the n^{th} degree represents a compound system of curves consisting of n sub-systems, each of these latter being such that the curves composing it do not meet one another. We will now point out the analogies and contrasts between resolvable and irresolvable equations in this respect.

Take for instance the equation

$$p^2y^2\cos^2\alpha-2pxy\sin^2\alpha+y^2-x^2\sin^2\alpha=0 \qquad \text{(A)}$$

which has already been discussed as regards the method of solution only in the previous article. We have seen that the complete primitive of this equation is $x^2+y^2-2kx+k^2\cos^2\alpha=0$ (i) when k is an arbitrary constant, and this represents a system of circles having their centres on the axis of x and enveloped by the two straight lines $x=\pm\tan\alpha.\ x$. If we take any point on the plane and substitute its co-ordinates in (i) we get two values of k, which determines *two* curves of the system of circles passing through this point. Thus looked at from this point of view this irresoluble differential equation of the second degree does not represent two separate sub-systems of non-intersecting curves, as a resoluble equation would represent, but one system only, which is such that two curves belonging to it pass through a given point.

But there is another point of view from which the system of curves represented by (i) may be looked at. We have seen that we can solve (A) by splitting up into two equations linear in p and then solving them independently. In fact the equation (A) is equivalent to two equations, *viz.*—

$$p=\frac{x\sin^2\alpha+\sqrt{x^2\sin^2\alpha-y^2\cos^2\alpha}}{y\cos^2\alpha}\quad\text{(B) and}$$

$$p=\frac{x\sin^2\alpha-\sqrt{x^2\sin^2\alpha-y^2\cos^2\alpha}}{y\cos^2\alpha}\qquad\text{(C)}$$

and we have seen that the solutions of (B) and (C) must be the equations linear in k obtained by solving (i), *viz.*—

$$k=\frac{x\pm\sqrt{x^2-(x^2+y^2)\cos^2\alpha}}{\cos^2\alpha}$$

$$\text{i.e.,}\quad x-k\cos^2\alpha=+\sqrt{x^2\sin^2\alpha-y^2\cos^2\alpha}\qquad\text{(ii)}$$

$$\text{and}\quad x-k\cos^2\alpha=-\sqrt{x^2\sin^2\alpha-y^2\cos^2\alpha}\qquad\text{(iii).}$$

Now if the equation (A) were a resolvable one, the differential equations B and C, or their complete primitives (ii) and (iii) would represent systems of non-intersecting curves and the equation itself would represent the compound system formed of the two subsidiary systems. Let us see if the same will hold in the case of this irresolvable equation also. For this it is only necessary to examine the geometrical equivalents of equations (ii) and (iii).

Whatever loci equations (ii) and (iii) may represent, the equation $(x-k\cos^2\alpha)^2=x^2\sin^2\alpha-y^2\cos^2\alpha$ will represent the two loci together. But this latter equation, as we have seen, represents for a particular value of k, a circle whose centre is at a distance k from the origin on the axis of x, and which touches the two lines $y=\pm\tan\alpha.x$. Hence the equations (ii) and (iii) must represent different parts of this circle. And it is evident that (ii) represents that portion of the circle for which $x-k\cos^2\alpha$ has got a positive value, and that (iii) represents that portion for which $x-k\cos^2\alpha$ is negative. Now if M, M' be the points where the enveloping lines $y=\pm\tan\alpha.x$ meets the circle, and if MM' meets the axis of x in K, then it is evident that $OK=k\cos^2\alpha$, and for the portion of the circle to the right of MM', $x-k\cos^2\alpha$ is positive, whereas for that portion which is to the left of MM the expression $x-k\cos^2\alpha$ is essentially a negative quantity.

Thus we come to the conclusion that the entire quadratic (i) represents the whole circle, and its two component equations, (ii) and (iii) taken together, represents the same thing. But equation (ii) alone represents the portion MBM', and the equation (iii) the portion MAM'.

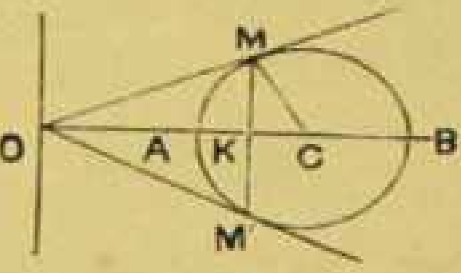

FIG. 10.

Now if in equation (iii) instead of considering k as a fixed constant we consider it to be arbitrary, the equation would represent a system of such circular arcs as MAM'. It will be seen easily by an attentive examination of the figure that this system of arcs may be described as a non-intersecting one, that is, one the members of which do not intersect one another. Similarly in (ii) k being considered arbitrary it represents a system of circular arcs like MBM', and this is also a non-intersecting system.

The fact that (ii) and (iii) represent non-intersecting systems also follows easily from an inspection of the differential equations of which they are solutions, *viz.*, equations (B) and (C), above. In these p at any point has got a single value, and just as in the case of equations considered in Art. 2, the curve systems represented must be non-intersecting.

Hence though differential equations of the first order which are not resolvable into rational and linear ones, seem at the first sight to represent one system of curves, two or more of which pass through any point and not two or more distinct non-intersecting systems, we may in one sense affirm that they do represent two or more systems of non-intersecting curves or rather parts of curves represented by rational equations.

We shall give one or two other examples. Take for instance the equation $xp^2 - 2yp + 4x = 0$ already considered in Art. 6. The solution of the equation is $x^2 = c(y - c)$ which represents a system of parabolas having the axis of y as axis and touching the straight lines $y = \pm 2x$. The solutions of the component equations are $y - 2c = \pm\sqrt{y^2 - 4x^2}$, and accordingly one of the equations in (i) represents the portion of the curve for which $y-2c$ is positive, and the other for which $y-2c$ is negative. Now $2c$ is the distance at which the polar of the origin with respect to the origin meets the axis of y. Hence the two equations represent non-intersecting systems as shown in the figures 11, 12 and 13.

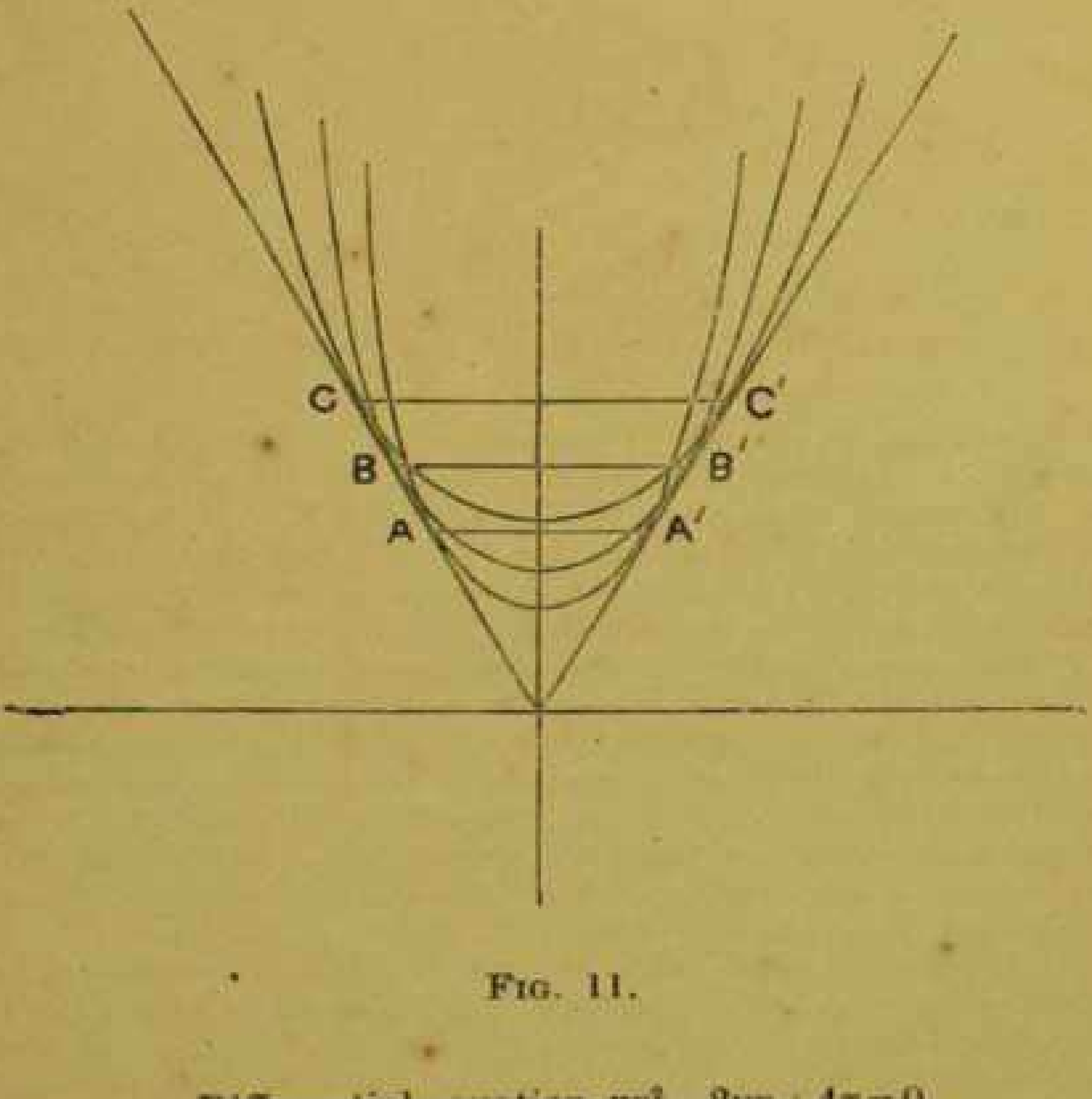

FIG. 11.

Differential equation $xp^2-2yp+4x=0$.
Complete primitive $x^2=c(y-c)$.

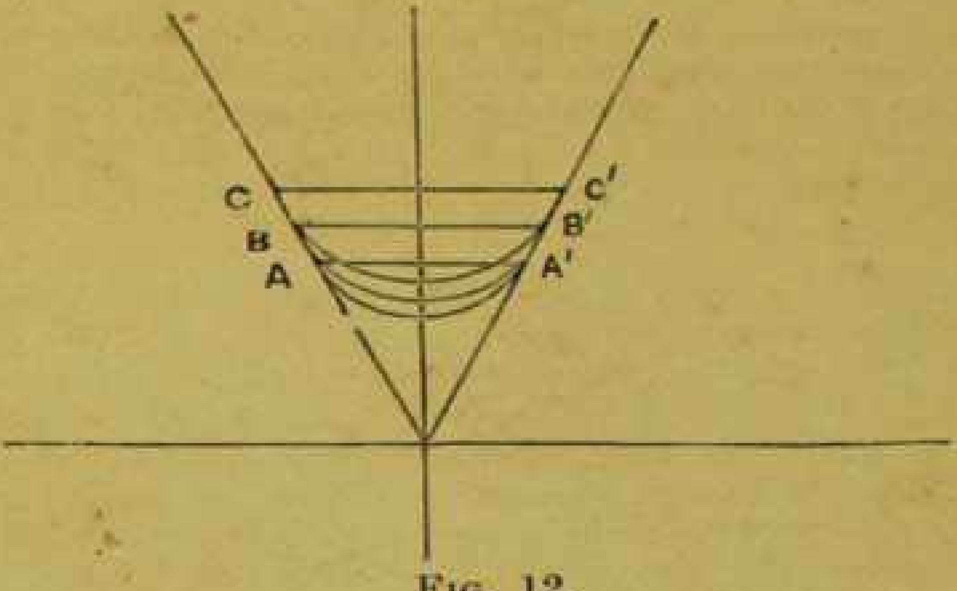

FIG. 12.

Differential equation $xp = y - \sqrt{y^2 - 4x^2}$.
Complete primitive $y - 2c = -\sqrt{y^2 - 4x^2}$.

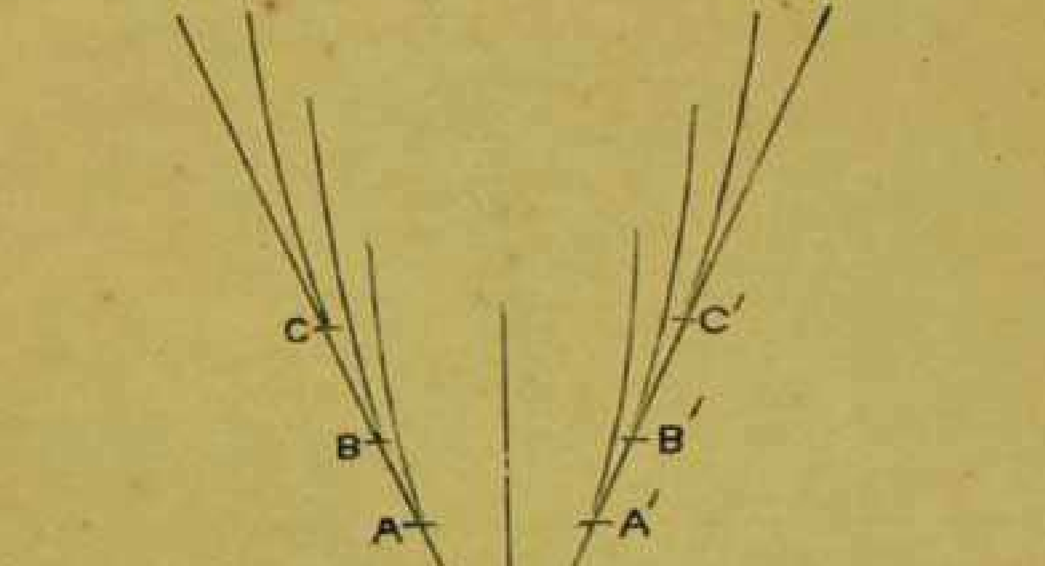

FIG. 13.

Differential equation $xp = y + \sqrt{y^2 - 4x^2}$.
Complete primitive $y - 2c = +\sqrt{y^2 - 4x^2}$.

Take another example $(x^2-a^2)p^2-2xyp-x^2=0$. This equation has been already solved in p. The complete primitive is $c^2+2cy+a^2-x^2=0$, which being put in the form

$$x^2=2c\left\{y+\frac{a^2+c^2}{2c}\right\}$$

shows that if c be a fixed constant the curve represented is a parabola having the axis of y for its axis. If c is arbitrary the equation represents a system of such parabolas enveloped by the circle $x^2+y^2-a^2=0$. The solutions of the component equation, as has been seen, is $y+c=\pm\sqrt{x^2+y^2-a^2}$. One of these, then, represents that part of the curve for which $y+c$ is positive; the other that part for which $y+c$ is negative. Now where the parabola $c^2+2cy+a^2-x^2=0$, and the circle $x^2+y^2-a^2=0$ touch one another, $y+c=0$. Hence above this point $y+c$ must be of one sign, below this point of another sign. Hence the two non-intersecting systems represented by the whole equation may be shown as in the figures 14, 15 and 16.

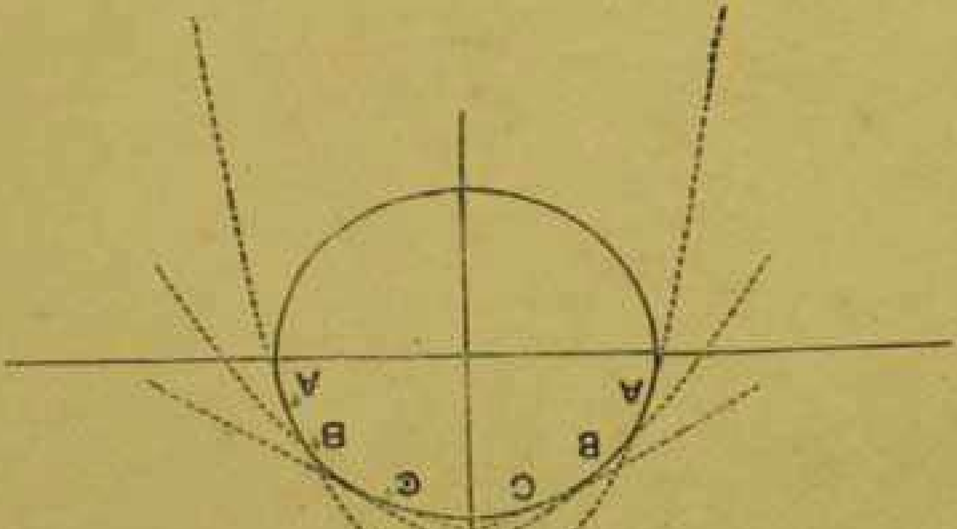

FIG. 14.

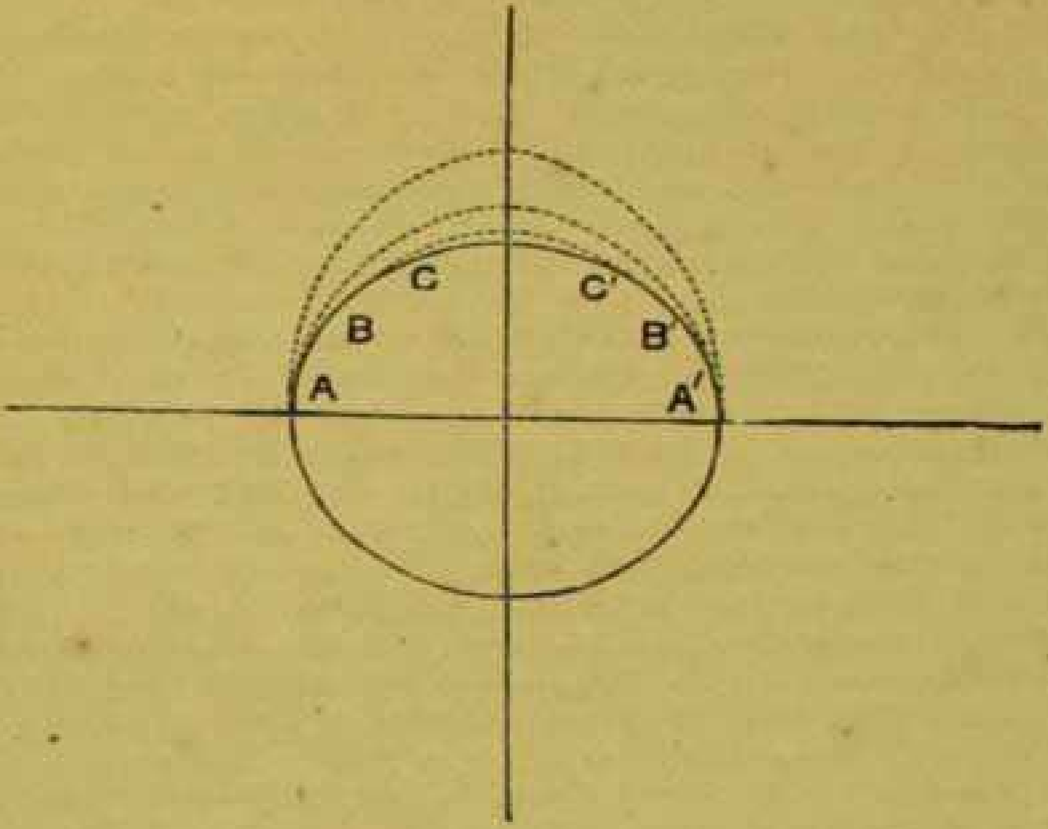

FIG. 15.

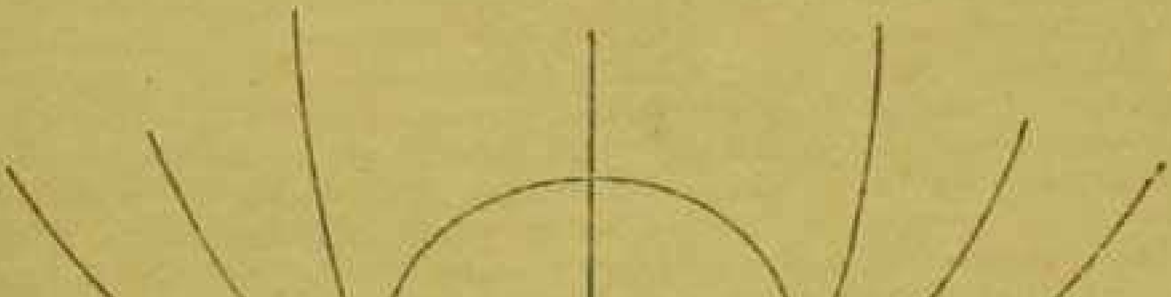

FIG. 16.

Thus it is clear that if we have an irresolvable differential equation of the first order and second degree, the complete primitive of it represents a system of curves such that two of them pass through one point. But by dividing each curve into two parts in a systematic way we can divide up the system of curves into two non-intersecting systems, and these correspond to each of the two component equations of the differential equations and their complete primitives exactly as in the case of resolvable equations.

The generalization of this theorem for equations of higher degree in p is difficult. On the one hand we have seen that an equation of the n^{th} degree in p can be resolved into n linear equations in p if we do not mind introducing irrational and imaginary quantities, and that to each of these component equations of the differential equations correspond a component of the complete primitive. The geometrical equivalent of a linear equation in p or its complete primitive being a system of non-intersecting curves, we see that the system of curves represented by the differential equation of the n^{th} degree or its complete primitive though at first appearing as one entire system of curves such that n of them passes through any point in the plane, may be looked upon, by dividing each curve into n parts, as made up of n sub-systems of curves, each of which in itself is non-intersecting. But on the other hand the algebraical solution of cubics and equations of higher degrees almost always introduces imaginary quantities, and this introduction of unreal magnitudes is a great obstacle in the way of clearly understanding the geometrical equivalent of each component of the differential equation. Thus in the case of the cubic $p^3 - 4xyp + 8y^2 = 0$ (i) we have found out the three component linear equations in p and their respective solutions. But though the system of curves represented by (i) and its complete primitive is perfectly determined, we cannot find out on what principle each curve of the system is to be divided into three parts such that we may get three sub-systems of non-intersecting curves corresponding to the component equations. But this much is certain, that if all the three component equations *have* any geometrical equivalent, some such method of dividing each curve must exist. But the question whether each component equation has a corresponding geometrical curve system has not yet been fully treated, and we are compelled to leave the question at this stage at present, not doubting that its fuller and more detailed investigation will lead to important results.

Article 8.

Singular Solutions.

Having seen in the two previous articles that in the method of solution no distinction need be made between resolvable and irresolvable equations, and that the analogy between these two classes of equations may be pushed even as regards their geometrical interpretation, we proceed to consider a point in which the analogy completely breaks down. We saw in Art. 6 that if the complete primitive of a differential equation consists of one system of non-intersecting curves we can ordinarily form no derived geometrical solution by taking a finite number of points on each of them: and if we have two or more such systems there can be no such derived *curve* in the ordinary sense of the term. The reason of the latter fact was not that a derived curve could not be conceived, but that when formed it turned out to be a curve such that its direction varied abruptly from one value to another at all points on it—a curve the study of which was impracticable from geometrical as well as from an analytical point of view. In the present case of irresolvable equations, however, passage from a curve of one system to a curve of another need not necessarily mean a sudden change of direction, for different parts of the same continuous curves are members of the two systems. Accordingly in this case, though the complete primitive consists, in one sense, of systems of non-intersecting curves, a derived geometrical solution of the nature of envelope can be formed.

In Art. 5 it was seen that of all the derived geometrical particular solutions of a non-linear differential equation of the first order, the envelope of the system of curves forming the geometrical complete primitive, if it can be found, is the only one that can be represented by an equation. From this singular property which it possesses among all the derived geometrical particular solutions, the name singular solution given to it is eminently appropriate. And though this view of looking at the envelope singular solution is novel, it is undoubtedly the one that makes the relation between a singular solution and the complete primitive the clearest possible. The singular solution is nothing but one of the derived geometrical particular solutions constructed in a particular way, *viz.*, by taking a finite number of points on each fundamented G.P.S., and it turns out on examination to be the only one that is analytically expressible.

A closer examination of the peculiar circumstances under which a system of curves consisting of non-intersecting subsystems is seen to have an envelope in the preceding paragraph may be interesting. Take the case discussed in Art. 7, the case of a system of circles enveloped by two intersecting right lines (Fig. 17). The way to look upon this system of circles as consisting of two non-intersecting systems of arcs is that shown in figs. 18 and 19. We will first of all examine the relation of the straight lines *OA*, *OB* to these subsystems.

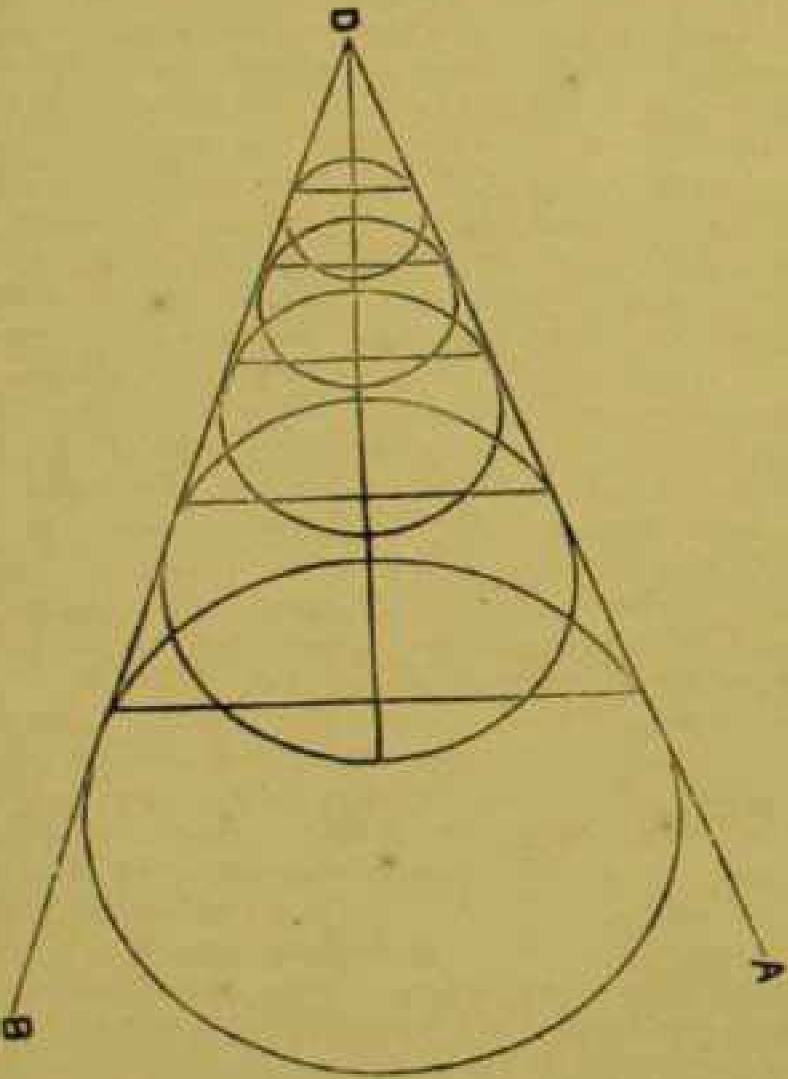

Fig. 17.

In the first place we ask, are these lines envelopes of the curves in fig. 19? The answer must be in the negative whether we look at the question from the geometrical or the analytical point of view. Geometrically OA, OB are not envelopes because an envelope is defined as the locus of ultimate points of intersection of the consecutive curves of a system, and this particular system of curves is non-intersecting. Analytically, the equation of the curve system is, as we have seen

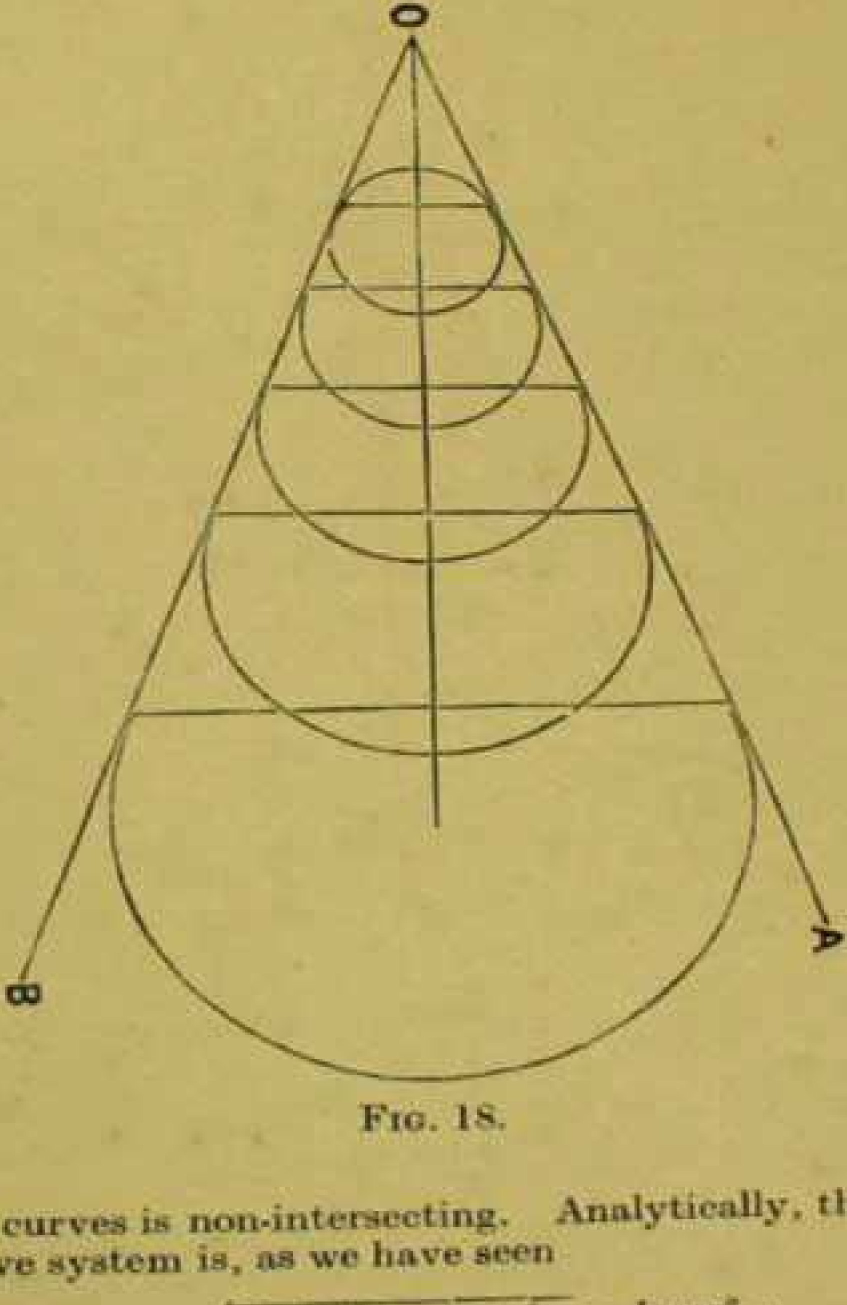

FIG. 18.

$$x + \sqrt{x^2 \sin^2 \alpha - y^2 \cos^2 \alpha} = k \cos^2 \alpha$$

and the ordinary process (*viz.*, differentiation on elimination of the arbitrary constant) of finding an envelope lead to no result in this case, as indeed we expect from the geometrical constitution of the system. Similarly in fig. 18 we see that the lines *OA*, *OB* are not the envelopes of the system of curves in the ordinary sense of the term.

But yet the lines envelope the system of circles taken as a whole, that is, the system of curves consisting of the two systems, each of which taken separately is not, as we have seen, enveloped by it. From a careful examination of the figures 17, 18 and 19, it is not difficult to realize how, though successive curves of the systems (18) and (19) do not intersect, those of system (17) do intersect, and give rise to an envelope.

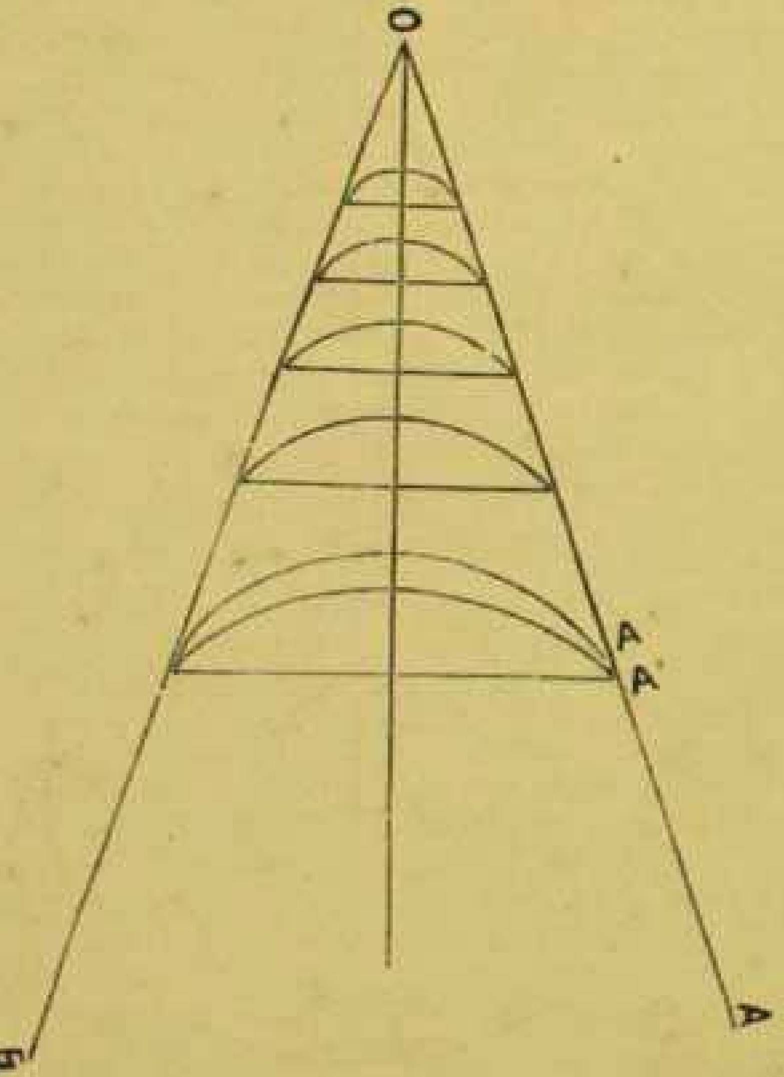

FIG. 19.

Having ascertained that the curves in fig. (19) are not enveloped by *OA*, *OB*, we ask, are *OA*, *OB* singular solutions of the differential equation of which the system (19) is the complete primitive. The answer to this question must be in the affirmative. For though in moving from A to the next point A' on the line OA, a point does not move in any of the curves of the system, yet it is moving in the direction of the tangent of the curve at that point, and consequently in accordance with the dictates of the differential equation. AA' is along the tangent of the curve through A and yet not along the curve, for the simple reason that the curve does not extend beyond A. The line OA is accordingly a solution of the differential equation and a singular solution since it is a derived curve made up of one point on each of the curves constituting the complete primitive. Similarly the line OB is a singular solution and OA, OB are also singular solutions of the system of curves in (18).

We will now turn back to what was stated in Art. 2. It was there said that if a differential equation represented a non intersecting system of curves a point could not pass from one curve of the system to another in accordance with the differential equation unless under exceptional circumstances. Now cases like the one above discussed are the exceptional cases referred to there ; for in that case we have a non-intersecting curve system, and the point describing the lines AO, OB passes through all the successive curves of the system, but still its motion is always in strict accordance with the value of $\dfrac{dy}{dx}$ given by the differential equation.

We proved in Art. 4 that a derived geometrical particular solution in order to be perfect (*i.e.*, analytically expressible) must consist of a finite number of points taken out of each fundamental geometrical particular solution. And since the direction of the derived particular solution must be identical with that of the fundamental G.P.S. from which that point is taken, two points must at least be in general taken from every fundamental G.P.S. in forming the perfect derived G.P.S. In the present case, however, we see that the perfect derived G.P.S. is made up of one point only on each of the fundamental geometrical particular solutions of the differential equation.

From what has been stated above it is clear that a differential equation may have a singular solution, though its geometrical complete primitive may be a non-intersecting curve system, and consequently not have an envelope defined as the locus of ultimate intersections. The singular solution must, however, always coincide in direction with some member or other of the G.C.P., and hence instead of saying that the singular solution is the envelope of the G.C.P. we prefer to call that it is the contact-curve of the G.C.P., *i.e.*, the curve which touches all members of the G.C.P.

In the above cases the equations were irrational ; but the same peculiarities will happen whenever we have a system of suddenly stopping curves, the points d'arret being so arranged in a line that the directions of the curves at any point d'arret is the direction of the line at that point. The locus will, in that case, be a singular solution, though not an envelope of the system of curves.

For example take the equation $\frac{dy}{dx} = \log x + 1$ the solution of which is $y = x \log x + c$.

This represents a series of suddenly stopping curves, each of which touches the axis of y in a point d'arret. Accordingly $x = 0$ is a singular solution of the differential equation though it is not an envelope of the system of complete primitives (fig. 20).

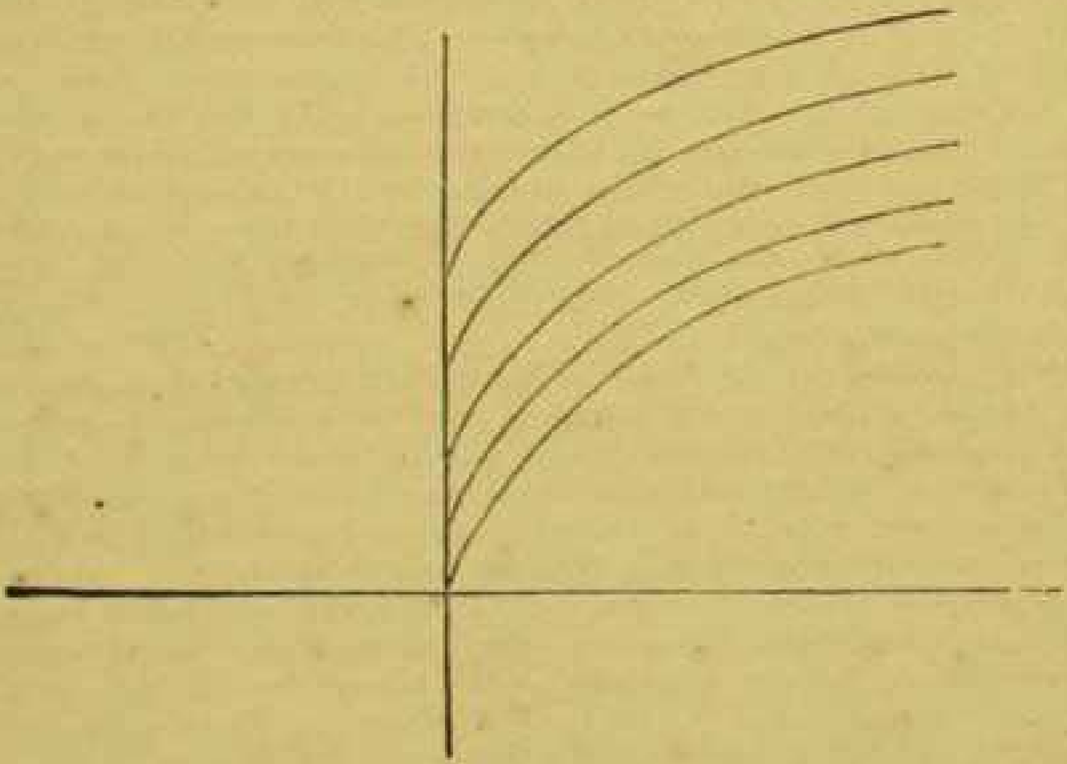

FIG. 20.

Article 9.

Analogy with Equations of Finite Differences.

Just as a differential equation of the first order is a relation between x, y and $\frac{dy}{dx}$, so a difference equation of the first order may be defined as a relation between x, u_x and Δu_x. Any relation between x and u_x is said to be a solution of the difference equation when on Δ being performed it leads to the same value of Δu_x, with the help of the original relation between x and Δu_x as is given by the difference equation.

A difference equation, like a differential equation, is capable of geometrical interpretation. Suppose we take any point x, u_x. Then the difference equation gives a value of Δu_x and we get the next point, whose co-ordinates are

$$x+1, \left\{ \begin{matrix} u_x + \Delta u_x \\ \text{or} \\ u_{x+1} \end{matrix} \right. .$$

Using these values again in the difference equation we get a value of Δu_{x+1} and we get to the next point whose co-ordinates are $x+2$, u_{x+2}. After r such operations we get at the r^{th} point $x+r$, u_{x+r}. And we see that if we start with any point in the plane we get an infinite series of points as the geometrical representation of the difference equation.

Any of these series of points is such that passage from one point to the consecutive point is in accordance with the dictates of the difference equation. Any such point series may be called a geometrical solution of the difference equation, and in contradistinction a functional relation between x and u_x leading to the same value of Δu_x as the given difference equation may be called an analytical solution.

If a difference equation is linear, and if there is no ambiguity of sign in any term, there is only one value of Δu_x for a given value of x and u_x and one set of values for $\Delta^2 u_x$, $\Delta^2 u_x$, etc. Accordingly it represents a system of point-series such that no two of them have one or more points common. For the co-ordinates of the n points of a series commencing with x, u_x being

$x+1$, u_{x+1}; $x+2$, u_{x+2}, etc., up to $x+n-1$, u_{x+n-1}
that is $x+1$, $u_x + \Delta u_x$

$$x+2,\ u_x + 2\Delta u_x + \Delta^2 u_x$$
$$x+3,\ u_x + 3\Delta u_x + 3\Delta^2 u_x + \Delta^2 u_x$$

$$x+n-1,\ u_x + (n-1).\ \Delta u_x + \frac{(n-1)^{(2)}}{\lfloor 2}\Delta^2 u_x + \ldots + \Delta^{n-1} u_x$$

the condition that n points of one point series may be identical with n points of any other point series and not more, is that u_x and its successive differences up to $n-1^{th}$ will be equal each to each, as given by the equations in the two series, but the subsequent differences are unequal. Thus in figure (21) the three point series *A*, *B* and *C* have two points *P* and *Q*, common to each other. Then whether we represent them by functional equations, *i.e.*, relations between x and u_x or by difference equations, *i.e.*, relations between x and u_x and Δu_x the values of u_x and Δu_{x-1} for the point *P* obtained from the equations of (*A*), (*B*) and (*C*) must be the same, but $\Delta^2 u_x$ and the higher differences will have different values. Similarly if as in figure (22) there are two point series, *A* and *B* having four points from P_1 common, then the value of u_x, Δu_x, $\Delta^2 u_x$ and $\Delta^3 u_x$ for

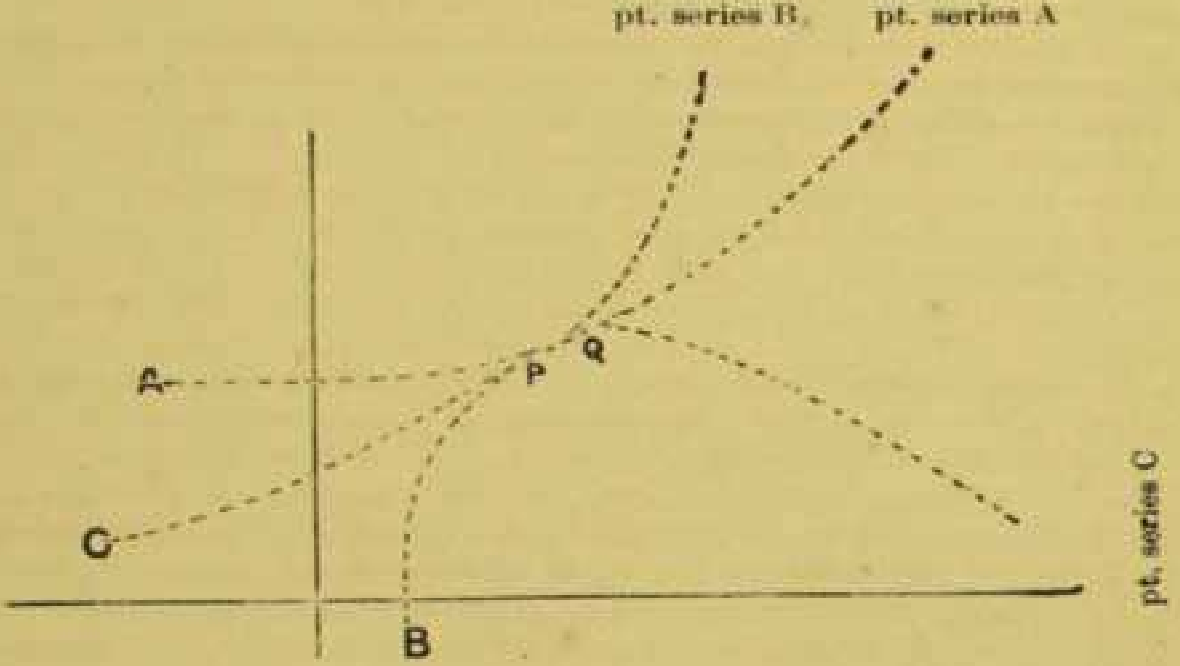

FIG. 21.

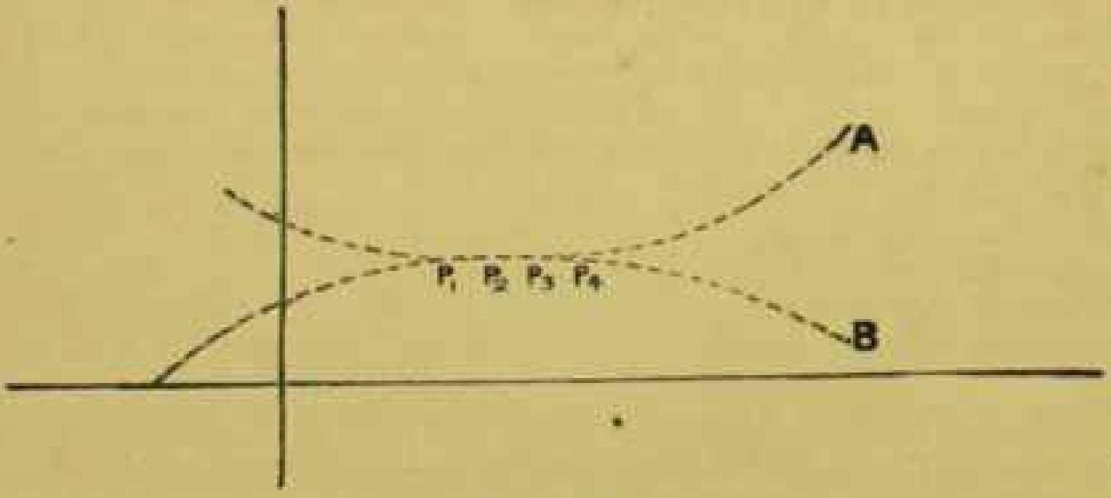

FIG. 22.

the point P_1, obtained from A and B, must be the same; but $\Delta^4 u_x$ and higher differences will be different.

From the above it is clear that the system of point-series represented by a rational difference equation of the first order and degree is non-intersecting, *i.e.*, no two of them have one or more points common to one another. For, if possible, let two of the series have a common point. Then since the same equation represents the two series the value of Δu_x, $\Delta^2 u_x$, etc., up to $\Delta^\infty u_x$ for that point are the same in both, *i.e.*, if the series have one common point all their points are common; that is, there are no two different intersecting series in the system which is represented by a linear difference equation of the first order.

Now if, as is usual in the calculus of finite differences, x is supposed to have unit increments, an equation between x and u_x represents a single point series. A singly infinite system of point series will accordingly be represented by an equation in x and u_x containing one arbitrary constant. Now we have seen that there is a singly infinite system of point series such that each of them is described according to the directions of a given difference equation. This system taken as a whole may be said to be the geometrical complete primitive, while the corresponding analytical relation between x, u_x and an indeterminate constant would be the analytical complete primitive.

As in the corresponding case of differential equations the geometrical solutions included in the G.C.P. are the only geometrical solutions of the equation, no solution can be framed by taking different parts of different point series. For a point moving in accordance with the difference equation can only trace one of these point series, it can never pass from one point series to another without violating the condition imposed by the difference equation.

Next suppose the difference equation to be rational and of the first order, but not of the first degree, *e.g.*, let it be $(\Delta u_x - a)(\Delta u_x - b) = 0$.

Then on taking the component equation $\Delta u_x - a = 0$ it is at once seen that the geometrical solution of the equation is a series of equidistant points arranged in a right line. This also becomes evident on solving the equation. For if $u_{x+1} - u_x = a$ we get $u_x = \Sigma \frac{a}{1} + c = ax + c$ (i), and if the co-ordinates of any number of points satisfy (i) it is evident that they are equidistant points situated on lines of the system $y = ax + c$. Similarly point series along any line of the system $y = bx + c$ are also solutions. And following the language employed in the case of differential equations we may look upon the geometrical solutions of the component equations taken together as the fundamental G. C. P. of the original compound equation.

The question which we now naturally ask is whether there are any derived G. S. in this case. And on referring to the figure it will be found that exactly the same facility exists in this case for the formation of derived geometrical particular solutions and G. C. P.'s in this case as in the corresponding case of differential equations. Thus we may be proceeding along any of the fundamental G.S.'s, but at any moment may turn to another fundamental G.S., and then again change our course after a time and so on, thus describing a point system which is not in itself a fundamental G. S., but every element of which is a part of some fundamental G. S.

Again in the case of differential equations corresponding to the class of difference equations now under consideration, we found that none of the infinite number of derived G. C. P's or derived G. P. S.'s were capable of analytical representation, and we naturally ask whether the same is true in this case also. In other words we will enquire whether any of the point series and systems of point series in figure (5) are capable of being represented by equations.

For instance take the system S_1 in that figure. Suppose the value of x from which the point system begins is r, an odd integer. Then the point system is seen to be formed thus; at each odd value of x, $\Delta u_x = a$, while corresponding to every even value of x, $\Delta u_x = b$, i.e., we may put in general for the difference equation of the point series $\Delta u_x + \frac{a+b}{2} + \frac{a-b}{2} (-1)^{x-r}$ and solving this we find for the functional equation of the system of point series

$$u_x = \left\{ \frac{a+b}{2} + \frac{a-b}{2} (-1)^{x-r} \right\} + c.$$

Thus we find that the system (i) of derived G. C. P. of equation (i) is capable of analytical representation. This is a very remarkable point of difference between the differential and difference equations of the class considered. Whereas no derived G. C. P. of $\left(\frac{dy}{dx} - a\right)\left(\frac{dy}{dx} - b\right) = 0$ is analytically expressible, we see there are derived G. C. P's of $(\Delta u_x - a)(\Delta u_x - b) = 0$ which can be as perfectly represented by equations in the fundamental system of G. C. P's.

We have now to consider the important point whether all G. P. S's and G. C. P's of (i), that can be constructed on the figure, have analytical equivalents like the one just discussed. For instance whether the system S_2 is so expressible.

Here we observe that when

$$\begin{array}{ll|ll} x=r & \Delta u_x = a & x=r+3 & \Delta u_x = a \\ x=r+1 & \Delta u_x = a & x=r+4 & \Delta u_x = a \\ x=r+2 & \Delta u_x = b & x=r+5 & \Delta u_x = b \end{array}$$

and so on in order. This curve will be analytically expressible if we can find a periodical functional expression for Δu_x in terms of a, b, x and r. In this case such a function would be

$$\Delta u_x = \frac{1+\omega^{x-r}+\omega^{2(x-r)}}{3}a + \frac{1+\omega^{(x-r)+1}+\omega^{2(x-r)+2}}{3}b$$

$$+\frac{1+\omega^{(x-r)+2}+\omega^{2(x-r)+4}}{3}a \qquad \text{(A).}$$

If this difference equation be solved we get the equation of the entire system of point series represented in figure (ii). Now equation (A) may be written,

$$3\Delta u_x = (2a+b) + \omega^{x-r}a + \omega^{2(x-r)}a + \omega^{x-r+1}b + \omega^{2(x-r+1)}b$$

$$+\omega^{x-r+2}a + \omega^{2(x-r+2)}a.$$

$$3u_x = (2a+b)x + \frac{\omega^{x-r}}{\omega-1}a + \frac{\omega^{2(x-r)}}{\omega^2-1}a$$

$$+\frac{\omega^{x-r+1}}{\omega-1}b + \frac{\omega^{2(x-r+1}}{\omega^2-1}b$$

$$+\frac{\omega^{x-r+2}}{\omega-1}a + \frac{\omega^{2(x-r+2)}}{\omega^2-1}a$$

$$= (2a+b)x + \frac{\omega^{x-r}}{\omega-1}\{a+\omega b+\omega^2 a\} + \frac{\omega^{2(x-r)}}{\omega^2-1}\{a+\omega^2 b+\omega a\}.$$

Generally speaking we know that if there are n quantities $a_1\ a_2\ a_3\ ..\ a_n$ and if these recur in any cyclical order, then a function can be found by using the n^{th} roots of unity, which assumes each of these values in succession as x passes through a series of values increasing by unity. For if α, β, γ etc. are the n^{th} roots of unity the expression $\dfrac{\alpha^x+\beta^x+\gamma^x \text{ etc.,}}{n}$ (i) is equal to unity if $x=n$ or a multiple of n, and is equal to zero, if x is any other quantity. And hence if the expression (i) is represented by P_x, the expression

$$P_{x-r}a_1 + P_{x-r-1}a_2 + P_{x-r-2}a_3 + \dots + P_{x-r-(n-1)}a_n =$$

E, is the expression we want. For evidently when $x=r, E=a_1$; when $x=r+1$, $E=a_2$ and so on until when $x=r+n$, $E=a_n$. Again when $x=r+n+1$ the value is a_1 for a second time, and so on cyclically as x increases continually.

Hence if in fig. (i) any point system is constructed in which for the values of Δu_x at successive points are a and b arranged in any cyclic order, however complicated, that point system will be capable of an analytical representation, and will accordingly be a perfect solution of the difference equation. Thus for instance suppose we have a point series, the Δu^x of which for successive points are—

$$a\ a\ b\ b\ b\ a\ a\ b\ b\ b\ a\ a\ b\ b\ b \text{ etc.,}$$

we may put

$$\Delta u_x = P_{x-r}\, a + P_{x-r-1} a + P_{x-r-2} b + P_{x-r-3} b + P_{x-r-4} b$$

where P_{x-r} etc. are expressed in terms of the fifth roots of unity. In short all G. P. S's which are repeating in their character (the portion which repeats being howsoever large and irregular), and all G. C. P's which are made of systems of such repeating point systems have equational equivalents of the form as illustrated before, and are accordingly perfect solutions of the difference equation.

If we want to further discuss whether any more of the geometrical solutions of the difference equation are perfect solutions, we are at once drawn into problems of theory of functions pure and simple. Given any G. S. we can write out the values which Δu_x has in it at successive values of x. We thus get a series of a's and b's following one another. If we can find a function such that as x passes through successive values differing by unity this function assumes the successive values of Δu_x in question, then we know that the G. S. is an analytical solution also. But if such a function cannot be found the G. S. will be an imperfect solution.

What we have said with respect to the simple resoluble equation $(\Delta u_x - a)\ (\Delta u_x - b) = 0$ applies equally well to any resoluble equation $\{\Delta u_x - f_1(x, u_x)\}\{\Delta u_x - f_2(x\ u_x)\} = 0$. The only difference will be that the fundamental G. S.'s will not be equidistant points arranged along right lines, but irregular point systems arranged along complicated curves.